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Volume of Trade and Dynamic Network Formation in Two-Sided Economies*

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Abstract

We study the long-run stability of trade networks in a two-sided economy. Each agent desires relationships with the other side, but having multiple partners is costly. This cost-benefit tradeoff results in each agent having a single-peaked utility over the number of partners-the volume of trade-, the peak being greater for agents on one side than those on the other. We propose a stochastic matching process in which self-interested agents form and sever links over time. Links can be added or deleted, sometimes simultaneously by a single agent. While the unperturbed process yields each pairwise stable network as an absorbing state, stochastic stability in two perturbed processes provides a significant refinement, leading respectively to egalitarian and anti-egalitarian pairwise stable networks. These distinct network configurations have implications for the concentration on each side of the market of a random information shock, which may affect structurally identical economies differently. The analysis captures stylized facts, related to herd behavior, market fragmentation, concentration and contagion asymmetry, in several two-sided economies. It also rationalizes long-run population imbalance between the two sides of certain buyer-seller markets.

Key words: *Two-sided economies, trade networks, pairwise stability, stochastic stability, herd behavior, fragmentation, concentration, contagion asymmetry.*

JEL Classification: C73, D01, D03, D85, F16, J00.

Résumé

Nous étudions la stabilité à long terme des réseaux commerciaux dans une économie biface. Chaque agent désire avoir des relations avec l'autre côté, mais avoir des partenaires multiples est coûteux. Ce compromis coût-bénéfice a pour résultat que chaque agent a une fonction d'utilité à sommet unique sur le nombre de partenaires -le volume du commerce -, le sommet étant plus grand pour les agents d'un côté que pour ceux de l'autre. Nous proposons un processus stochastique de jumelage dans lequel les agents égoïstes forment et rompent les liens avec le temps. Les liens peuvent être ajoutés ou supprimés, parfois en même temps par un seul agent. Alors que le processus non perturbé rend chaque réseau apparié stable en tant qu'état absorbant, la stabilité stochastique dans deux processus perturbés fournit une amélioration significative, menant respectivement à des réseaux appariés stables égalitaires et anti-égalitaires. Ces configurations de réseau distinctes ont des implications sur la concentration d'information dans chaque côté du marché suite à un choc aléatoire, celui-ci étant aussi susceptible d'affecter différemment des économies structurellement identiques. L'analyse rend compte de faits stylisés liés au comportement grégaire, à la fragmentation et la concentration du marché et à l'asymétrie de contagion dans plusieurs économies bifaces. Elle rationalise aussi le déséquilibre à long terme des populations entre les deux côtés de certains marchés d'acheteurs et de vendeurs.

Mots clés : *Économie biface, réseaux commerciaux, réseaux appariés stables, stabilité stochastique, comportement grégaire, fragmentation, concentration, asymétrie de contagion.*

Classification JEL : C73, D01, D03, D85, F16, J00.

1 Introduction

We propose a dynamic theory of network formation in a two-sided economy. Although the model covers several applications, where the sides of the market could be rich/poor countries, buyers/sellers, employers/workers, faculty/students, and so on, to fix ideas, we shall refer to agents on each of both sides as *men* and *women*, respectively. Each agent derives utility from having relationships with the other type. However, having many partners is costly. This cost-benefit tradeoff results in each agent having a single-peaked utility function. Preference heterogeneity is possible, but we assume peak-homogeneity within each side of the economy. Moreover, our key maintained assumption is that the peak (or optimal number of partners) is greater for men than for women (i.e., one side of the market wishes to have more links with the other side).

A link between a man and a woman can be viewed as representing a *trade* relationship in which the two parties exchange "goods" of some sort. A good might be material, as in a real exchange market, or emotional, such as between lovers. Agents derive utility only from the number of partners they have, which we call the *volume of trade*. Although each relationship involves an exchange of goods, we choose to model network formation as in a repugnant market (Roth (2007)) where links cannot be purchased or sold.

Our goal is three-fold. First, we provide a full characterization of statically stable networks. Second, building on this characterization, we extend the analysis to a dynamic setting, yielding a characterization of networks that arise in the very long run. Third, we apply our findings to capture stylized facts, related to herd behavior, market fragmentation, and concentration, in several two-sided economies. Also, we apply a new index of communication and contagion to the long-run networks to study asymmetry in the concentration of a random information shock.[1]

The current analysis contributes to two broad literatures: the literature on two-sided matching (with capacity constraints), and the literature on endogenous network formation. Its distinctive feature, however, is that agents in our model do not direct their links but decide the number of partners (or the volume of trade). Here, the link formation process is not equivalent to elaborating a nominal list of intended relationships, as is the case in these literatures.[2] This simplification enables us to totally characterize statically and long-run stable matchings in terms of the allocation of links between partners and to study their diffusion properties for each side of the economy.

Our model may be used in a wide variety of contexts where agents care mostly about the number of partners they have. Fidelity economies are an example (Pongou (2010)). In these economies, having many partners of the opposite type may be viewed as infidelity, which is punished if detected. An example of such a market is

[1] The nature of the shock depends on the type of the economy. For instance, in a faculty-student economy, a shock might be a new idea or research question. In a dating economy, a shock might be a new sex technology or a sexually transmitted disease.

[2] Although agents might differ in various characteristics in a two-sided market with buyers and sellers, for instance, we assume that these characteristics do not enter their potential partners' utility functions. A chocolate seller might not care about whether his buyers are tall or short, white, black, or purple: he only cares about the number of sales.

an employer-employee market where a contract stipulates a *prima facie* duty of loyalty of an employee to her employer, which precludes working for a competing firm. Here, the requirement of fidelity is only one-sided. Another example of a fidelity economy is the dating market (Pongou and Serrano (2013)). Indeed, Pongou and Serrano (2013) apply some of the insights from the current paper in order to understand the mechanisms driving the long-run gender differences in HIV/AIDS prevalence across societies.

The model has several other applications, including buyers and sellers transacting in a market for a continuous good. In this case, one can argue that the optimal number of buyers for each seller exceeds the optimal number of stores each buyer purchases from. Another example is the international market between rich and poor countries, where the optimal number of trading partners is generally greater for the former than for the latter. Yet another example is the economy involving graduate students writing a doctoral dissertation and their faculty advisors. Here, the number of optimal links for students (the size of the doctoral committee) is usually lower than it is for professors (the number of committees they can join). Our model provides a rationale for understanding why, in most of these economies, we may see long-run situations in which the number of active agents on one side may be smaller than the number of agents on the opposite side.

In what follows, we provide an overview of the findings, shedding light on the patterns of relationships that form in two-sided economies such as the ones just mentioned, and showing how these relationships subsequently affect the spread of information, with likely different outcomes for each side and across structurally identical economies.

1.1 Static Analysis of Two-Sided Networks: Pairwise Stability

We fully characterize the *pairwise stable networks*. In a matching problem such as ours, individuals form new links or sever existing ones based on the reward that the resulting network offers them relative to the current network. For simplicity, as in most of the literature, agents are myopic. We say that a network is pairwise stable if: (i) no individual has an incentive to sever an existing link in which he or she is involved; and (ii) no pair of a man and a woman have a strict incentive to form a new link between them while at the same time possibly severing some of the existing links they are involved in.[3]

Our characterization of pairwise stable networks depends on the difference in the total supply of relationships between the two sides of an economy. In general, agents from the short side typically obtain their optimal number of partners, whereas agents from the long side obtain at most their optimal number of partners. The intuition behind this finding is straightforward. Agents from the short side supply a smaller number of links than the ones demanded by agents from the opposite side, which in turn results in only the latter competing for the former. However, and in addition to this intuitive finding, it might happen that, despite such competition for links, certain agents on the short side do not obtain their optimal number of partners. This is a situation

[3]See Gale and Shapley (1962) for a first use of pairwise stability. Within networks, Jackson and Wolinsky (1996) provide the standard definition. Our definition is slightly different: while they allow weak blocking in the pair, we assume strict blocking, in part due to the absence of side-payments.

of an inefficient allocation of links, perhaps an unexpected result at first blush. We show that this type of inefficiency is resolved under certain natural conditions related to the size of the economy. In particular, we show that, if the number of men is equal to that of women and the population is sufficiently large relative to the optimal numbers of partners, then only efficient pairwise stable networks arise. In such economies, a network is pairwise stable if and only if each woman has exactly her optimal number of partners, and each man has at most his optimal number of partners.

1.2 Dynamic Analysis: Steady-State Networks, Stochastic Stability, and Contagion

The center of our analysis is a dynamic matching process for the matching problem, more precisely a Markov process.[4] Random encounters between men and women are based on the incentives that agents have to form new links or sever existing ones. Specifically, the *unperturbed Markov process* assumes discrete time, and is defined as follows. In each period, a man and a woman chosen at random with arbitrary positive probability are given the opportunity to sever or add a link based on the improvement that the resulting network offers to each of them relative to the current network. If they are already linked in the current network, the decision is whether to sever the link; severance is a unilateral decision. Otherwise, the decision is whether to form a new link; link formation is a bilateral decision. While forming a new link, each agent is allowed to sever as many of the links he/she is involved in as possible in the current network (although, because of our single-peaked preferences assumption, without loss of generality, one can restrict attention to the case of severing only one link).

Under the same general assumptions that allow us to characterize pairwise stable networks in all our settings, we characterize the *long-run predictions* –steady or recurrent states– of this process. We find that the long-run predictions coincide with the set of pairwise stable networks, a very large set.

To gain predictive power in our analysis, the matching process is perturbed in two different ways, corresponding to two different *market cultures* and *perceptions* of multiple partnerships. Each perturbation consists of allowing a small probability of forming new links or severing existing ones when this action is not beneficial to the agents involved. Under the additional simplifying assumptions of the two sides having equal size and the populations being sufficiently large with respect to the optimal numbers of partners, we study the long-run predictions of these *perturbed processes* –their *stochastically stable networks* –, these predictions being the only networks that are visited a positive proportion of time in the very long run.[5]

[4] Jackson and watts (2002) also provide a related analysis of dynamic networks. As we discuss later, the two models have significant differences.

[5] In a perturbed process, one can no longer speak of "steady states," as by definition, there is always a positive probability of transiting from any state to any other. The notion of stochastic stability (Freidlin and Wentzell (1984)) provides a useful methodology to identify those states in which the perturbed process spends most of its time in the long run. It has been applied to study a number of problems in the economics literature (see, e.g., Foster and Young (1990), Kandori, Mailath and Rob (1993), Young (1993) for early contributions). Young (1998) presents many of its applications. The main shortcoming of stochastic stability is its associated slow speed of convergence, but it is very helpful in identifying long-run trends, our main interest here. Also, the reader should keep in mind that the frequency of a random encounter between a man and a woman may be extremely high, perhaps every minute depending on the type of market being considered, thus allaying the issue.

The perturbed processes allow for more transitions than their unperturbed baseline.[6] Specifically, in both processes, an action that worsens its initiator, which we shall call a *mistake*, occurs with a small probability $\varepsilon > 0$. Key to our analysis are actions that are neither beneficial nor mistakes, i.e., actions that leave their initiators exactly indifferent. We shall refer to these as *utility neutral actions* or *neutral actions* for short. These correspond to situations in which an agent severs an existing link with a current partner and forms a new link with another agent. We shall assume that the probability of taking such a neutral action is $\varepsilon^{f(\cdot)} > \varepsilon$ (because $0 < f(\cdot) < 1$). The exponent $f(\cdot)$ is the "*perceived*" *strength of the existing link* so that, in evaluating the merits of taking a neutral action, links that are perceived as stronger $-f(\cdot)$ closer to 1$-$ are harder to break.

We study two perturbed processes. In the first one, the strength $f(\cdot)$ of a link to be severed is *inversely* proportional to the number of partners that the partner to be abandoned has in the existing network. One interpretation is that this link is perceived as strong as the amount of time invested in it by that partner. We find that networks are stochastically stable if and only if they are *egalitarian pairwise stable networks.* Men and women have the same number of partners, which is the optimal number of partners for women. In contrast, the second perturbed process assumes that the strength $f(\cdot)$ of a link to be severed is *directly* proportional to the number of partners that the partner to be left behind has in the existing network. The individual who invests more time in a relationship either shows relative weakness to his/her partner, or (perhaps incorrectly) signals the "high quality" of the latter to other agents. For this case, we find that *anti-egalitarian pairwise stable networks*, which are networks in which each woman has her optimal number of partners, and the smallest possible set of men is matched, will be the only ones visited a positive proportion of time in the very long run. Each non-isolated man is matched to his optimal number of partners (except for at most one matched man, who will be matched to the remaining women). The rest of men will remain isolated.

We shall later describe several interpretations of the two processes in a number of real-life economies, providing a rationale for patterns of market fragmentation, herding, and concentration that are generally observed. A particular implication of the second process is that, in two-sided markets, the number of agents active in trade on one side may be much smaller than the number of active agents on the other side, such as in most buyer-seller markets. Here, herd behavior among buyers first selects, from an initially larger pool of sellers, those who remain in the market in the long run; after this selection has taken place, buyers match with the lucky sellers following a pattern of perfect competition, as each seller generally has an equal share of the market. In contrast, economies characterized by the first stochastic process do not eliminate any seller from trading in the market, with each seller obtaining an equal share, also reflecting a pattern of perfect competition. There are several advantages to considering both perturbed processes.[7] Theoretically, studying polar opposites in the assumptions behind neutral actions offers a more complete understanding of

[6] As it turns out, using uniform perturbations (e.g., as in Jackson and Watts (2002)), does not serve as a useful refinement in our model. Thus, we are led to explore other possibilities, which we motivate with extra-economic elements in society.

[7] Bergin and Lipman (1996) show that one can always construct processes with state-dependent perturbations that will select any subset of the steady states as stochastically stable. An important implication of this result is that one should motivate the particular perturbed processes that one chooses to work with.

the problem, and the rationale behind these actions offers an interesting alternative to justify the perturbations (in addition to mutations, experimentation or mistakes, invoked in previous literature). Empirically, the two approaches are consistent with different perceptions or sociological realities, prevalent in different societies or market cultures.

In a two-sided economy, the question of which side of the economy is more likely to be more affected by the spread of a random *unanticipated* information shock is important. In order to answer this question, we consider a simple index of contagion, and using it in our sharp results derived thanks to stochastic stability, we measure the proportion of each side that ends up receiving the shock; see Subsection 5.2 for details.

Indeed, we show that under the first perturbed dynamic process, the difference in contagion potential between the two sides of an economy in any of the stochastically stable networks is zero. Under the second process, women's contagion potential is always greater than men's in the long run.[8] We also find that information prevalence might differ across two structurally identical economies, a result of there being multiple equilibrium networks under each of the two stochastic processes.

1.3 Plan of the Paper

The remaining of this paper unfolds as follows. Section 2 introduces the model that forms the basis for our analysis. We characterize pairwise stable networks in Section 3. In Section 4, we analyze the dynamic process. In Section 5, we examine some applications of our model in several two-sided economies and study the implications for contagion asymmetry, as well as discuss related literature. Section 6 concludes.

2 The Model

The economy consists of a finite set of individuals $N = \{1, 2, \ldots, n\}$, partitioned into two sets of agents labelled *men* (M) and *women* (W), respectively. Each individual derives utility from direct links with opposite type agents. Engaging in multiple links is costly. This tradeoff results in each agent having a single-peaked utility function.

2.1 Utility Functions

A network g is a subset of $M \cup \{\emptyset\} \times W \cup \{\emptyset\}$, where $(m, \emptyset) \in g$ means that man m is isolated or has no connection in g, and similarly, $(\emptyset, w) \in g$ means that woman w has no connection in g.[9]

Let g be a network. Since we are dealing with undirected graphs, if $(i, j) \in g$, we will abuse notation and

[8] This result implies that women in a heterosexual economy, and students in an instructor-student economy are more affected by the spread of a new information than the other side. This result for the stochastically stable networks of the second process may seem surprising, given that the definition of the perturbed process itself is "gender neutral". However, in combination with our assumption of asymmetry in optimal numbers of partners across both sides, all the key transitions involve a woman severing a link to form a new one, and in doing so, the cost of breaking that link is a direct function of the dominant role of her former male partner, measured by the number of his links.

[9] More formally, for any man $m \in M$ such that $(m, \emptyset) \in g$, $(m, w) \notin g$ for all woman $w \in W$; similarly, for any woman $w \in W$ such that $(\emptyset, w) \in g$, $(m, w) \notin g$ for all man $m \in M$.

consider that $(j,i) \in g$ (in fact, (i,j) and (j,i) represent the same relationship). Let $i \in N$ be an individual, and $s_i(g)$ the number of opposite type partners that i has in the network g. The utility that i derives from g, denoted by $u_i(s_i(g))$, is assumed to be single-peaked. This utility function reflects a tradeoff between the benefit and the cost of forming links. Although each agent might have a different utility function, for simplicity, we assume peak-homogeneity within each side of the economy. Let s^*_m and s^*_w denote the peaks or optimal numbers of partners for each man and each woman, respectively. We assume that the peak is greater for men than for women ($s^*_m > s^*_w$), which reflects the fact that men desire more links than women, or that the cost of forming a link is greater for women than for men. We will call (M, W, s^*_m, s^*_w) a two-sided economy.

2.2 Definitions of Concepts in Networks

Let g be a network. The elements of N are called vertices. A path in g connecting two vertices i_1 and i_n is a set of distinct nodes (a node is a link between two individuals) in $\{i_1, i_2, \ldots, i_n\} \subset N$ such that for any k, $1 \leq k \leq n-1$, $(i_k, i_{k+1}) \in g$.

Let i be an individual. We denote by $g(i) = \{j \in N : (i,j) \in g\}$ the set of individuals who have i as a partner in the network g. The cardinality of $g(i)$ is called the degree of i. If a set A is included either in M or W, then the image of A (or the set comprising all the partners of agents in A) in the network g is denoted by $g(A) = \bigcup_{i \in A} g(i)$.

We denote respectively by $M(g) = \{i \in M : \exists j \in W, (i,j) \in g\}$ and by $W(g) = \{i \in W : \exists j \in M, (i,j) \in g\}$ the set of men and women who are matched in the network g. We pose $N(g) = M(g) \cup W(g)$.

A subgraph $g' \subset g$ is a component of g if for any $i \in N(g')$ and $j \in N(g')$ such that $i \neq j$, there is a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$ such that $(i,j) \in g$, $(i,j) \in g'$. Intuitively, a component of g is a maximal subset of directly or indirectly connected agents in g. A network g can always be partitioned into its components. This means that if $C(g)$ is the set of all the components of g, then $g = \bigcup_{g' \in C(g)} g'$, and for any $g' \in C(g)$ and $g'' \in C(g)$, $g' \cap g'' = \emptyset$ (i.e., two distinct components share no agents).

3 Pairwise Stable Networks

In an economy such as the one we are describing, individuals form new links or sever existing links based on the improvement that the resulting network offers them relative to the current network. We say that a network g is *pairwise stable* if: (i) no individual has an incentive to sever an existing link in which he/she is involved; and (ii) no pair of a man and a woman have an incentive to form a new link between them while possibly at the same time severing some of the existing links in which they are involved.

More formally, given a profile of utility functions $u = (u_i)_{i \in N}$, a network g is pairwise stable with respect to u if:

(i) $\forall i \in N$, $\forall (i,j) \in g$, $u_i(s_i(g)) \geq u_i(s_i(g \setminus \{(i,j)\}))$; and

(ii) $\forall (i,j) \in (M \times W) \setminus g$, if network g' is obtained from g by adding the link (i,j) and possibly severing other links involving i or j, $u_i(s_i(g')) > u_i(s_i(g)) \Longrightarrow u_j(s_j(g')) \leq u_j(s_j(g))$ and $u_j(s_j(g')) > u_j(s_j(g')) \Longrightarrow u_i(s_i(g')) \leq u_i(s_i(g))$.

According to (ii), (i,j) is a blocking pair whenever the two parties involved strictly benefit from the union. In this sense, link formation is driven only by self-interest, and so, an agent does not enter a relationship if he/she has no incentives to do so.[10] In this, our definition is different from the one introduced by Jackson and Wolinsky (1996), where two agents form a link if one is willing to do so and the other is indifferent.

To illustrate this definition, consider the following examples. A network in which a woman is matched to $s > s_w^*$ men is not pairwise stable as she can unilaterally sever $s - s_w^*$ links. A network in which a man is matched to $s_m^* + 2$ women and a woman not matched to him is matched to fewer than s_w^* men is not stable, as they could form a link while the man could sever three of his former links (alternatively, the man alone could sever only one of his links). Finally, a network in which a man and a woman who are unmatched have fewer than their optimal number of partners is not pairwise stable either, as they could form a link without severing any other.

We provide a full characterization of pairwise stable networks in Theorem 1 below:

Theorem 1 *Let $\mathcal{E} = (M, W, s_m^*, s_w^*)$ be an economy and g a network. Define the sets $A(g) = \{w : s_w(g) \neq s_w^*\}$ and $B(g) = \{m : s_m(g) \neq s_m^*\}$.*

PART I: *Suppose $|M|s_m^* < |W|s_w^*$.*

1) Suppose $|W| \leq s_m^*$. Then, g is pairwise stable if and only if $\forall m \in M,\ \ s_m = |W|$ and $\forall w \in W$, $s_w = |M|$.

2) Suppose $|W| > s_m^*$. Then, g is pairwise stable if and only if:

2-(a) $B(g) = \emptyset$ and $\forall w \in W$, $0 \leq s_w \leq s_w^*$; or

2-(b) $A(g) \neq \emptyset$, $B(g) \neq \emptyset$, and for any $m_1 \in B(g)$ and $w_1 \in A(g)$:

(i) $(m_1, w_1) \in g$;

(ii) $|A| \leq s_{m_1} \leq s_m^* - 1$;

(iii) $|B| \leq s_{w_1} \leq s_w^* - 1$.

PART II: *Suppose $|M|s_m^* = |W|s_w^*$.*

1) Suppose $|W| \leq s_m^*$. Then, g is pairwise stable if and only if $\forall m \in M,\ \ s_m = |W|$ and $\forall w \in W$, $s_w = |M|$.

2) Suppose $|W| > s_m^*$. Then, g is pairwise stable if and only if:

2-(a) $A(g) = B(g) = \emptyset$; or

2-(b) $A(g) \neq \emptyset$, $B(g) \neq \emptyset$, and for any $m_1 \in B(g)$and $w_1 \in A(g)$:

[10] In the absence of side payments, the strict improvement of each individual in the pair is a natural assumption (see, e.g., Aumann (1959)).

(i) $(m_1, w_1) \in g$*;*

(ii) $|A| \leq s_{m_1} \leq s_m^* - 1$*;*

(iii) $|B| \leq s_{w_1} \leq s_w^* - 1$*.*

PART III: *Suppose* $|M|s_m^* > |W|s_w^*$*.*

1) Suppose $|M| \leq s_w^*$*. Then, g is pairwise stable if and only if* $\forall m \in M$*,* $s_m = |W|$ *and* $\forall w \in W$*,* $s_w = |M|$*.*

2) Suppose $|M| > s_w^*$*. Then, g is pairwise stable if and only if:*

2-(a) $A(g) = \emptyset$ *and* $\forall m \in M$*,* $0 \leq s_m \leq s_m^*$*; or*

2-(b) $A(g) \neq \emptyset$*,* $B(g) \neq \emptyset$*, and for any* $m_1 \in B(g)$*and* $w_1 \in A(g)$*:*

(i) $(m_1, w_1) \in g$*;*

(ii) $|A| \leq s_{m_1} \leq s_m^* - 1$*;*

(iii) $|B| \leq s_{w_1} \leq s_w^* - 1$*.*

Proof. Let us first show:

Fact 1: If g is a pairwise stable network, then $\forall m \in M$, $0 \leq s_m \leq s_m^*$ and $\forall W \in W$, $0 \leq s_w \leq s_w^*$.

Indeed, if an agent has more than his/her optimal number of partners, he/she will be better off by unilaterally severing one link, thus implying that g is not pairwise stable, a contradiction.

PART I: Suppose $|M|s_m^* < |W|s_w^*$.

1) Suppose $|W| \leq s_m^*$. Therefore, $|M|s_m^* < s_m^* s_w^*$, i.e., $|M| < s_w^*$. Let g be a pairwise stable network.

Assume by contradiction that there exists a man m_0 such that $s_{m_0} < |W|$. Hence, $s_{m_0} < s_m^*$.

It follows that there exists a woman w_0 not matched with m_0. Thus, $s_{w_0} < |M| \leq s_w^*$.

Thus, m_0 and w_0 could form a link and improve, contradicting that g is pairwise stable.

We conclude that any man m has $s_m = |W|$ partners. This means that each man is matched to all the women, which also implies that each woman is matched to all the men. So $s_w = |M|$ for any $w \in W$.

Conversely, assume that $\forall m \in M$, $s_m = |W|$ and $\forall w \in W$, $s_w = |M|$. Since $|M| < s_w^*$ and $|W| \leq s_m^*$, no individual can improve by severing a link since he/she is at the upward sloping part of his/her utility function. Also, no individual can form a link with an opposite-type individual since he/she is already matched to all the opposite-type individuals. Therefore, g is a pairwise stable network.

2) Suppose $|W| > s_m^*$. Let g be a pairwise stable network. Suppose that 2-(a) is not true. We should therefore prove that 2-(b) is true. Recalling Fact 1, given that 2-(a) fails, there exists a man m_0 with $s_{m_0} < s_m^*$, i.e., $B(g) \neq \emptyset$. This also implies:

Fact 2. The set $A(g) \neq \emptyset$: if all women had s_w^* partners, the number of links supplied by women in g would be $|W|s_w^*$, which, by assumption, exceeds the optimal number of links that men can supply, a contradiction.

It remains to show that for any $m_1 \in B(g)$ and $w_1 \in A(g)$, assertions (i)-(iii) are true.

Assertion (i) is true because, since any man in $m_1 \in B(g)$ has fewer than his optimal number of partners and any woman $w_1 \in A(g)$ has fewer than her optimal number of partners, any such man and woman would have an incentive to form a link were they not matched.

To prove assertion (ii), let $m_1 \in B(g)$. The fact that $|A(g)| \leq s_{m_1}$ follows from assertion (i). The fact that $s_{m_1} \leq s_m^* - 1$ follows from the definition of $B(g)$.

The proof of assertion (iii) is similar to that of (ii).

Therefore, if g is pairwise stable and 2-(a) fails, 2-(b) must hold.

Now suppose that g is pairwise stable and 2-(b) fails. We therefore have to show that 2-(a) holds. Recalling Fact 1, it only remains to show that $B(g) = \emptyset$. By contradiction, if $B(g) \neq \emptyset$, and recalling Facts 1 and 2, we would have also that $A(g) \neq \emptyset$. Furthermore, we could easily establish assertions (i), (ii), and (iii), leading to 2-(b) holding, contradicting our supposition. Hence, $B(g) = \emptyset$ and 2-(a) must hold.

We now prove that, if 2-(a) or 2-(b) is true, then g is pairwise stable. Suppose that 2-(a) is true. A woman alone cannot improve by severing a link since she is at the upward sloping part of her utility function. She cannot form a new link with another man since each man has his optimal number of partners. And a man cannot be part of any blocking move (either by himself or with a woman) since he is at his peak. Therefore, g is a pairwise stable network.

Suppose that 2-(b) is true. A man in $B(g)$ alone cannot improve by severing a link since he is at the upward sloping part of his utility function. He cannot form a new link with another woman in $A(g)$ since he is already matched to all the women in $A(g)$. He cannot form a new link with another woman in $W \setminus A(g)$ since each woman in $W \setminus A(g)$ has her optimal number of partners. A man in $M \setminus B(g)$ cannot improve by severing or forming a link since he is at his peak.

Similarly, a woman in $A(g)$ alone cannot improve by severing a link since she is at the upward sloping part of her utility function. She cannot form a new link with another man in $B(g)$ since she is already matched to all the men in $B(g)$. She cannot form a new link with another man in $M \setminus B(g)$ since each man in $M \setminus B(g)$ has his optimal number of partners. A woman in $W \setminus A(g)$ cannot improve by severing or forming a link since she is at her peak. Therefore, g is a pairwise stable network.

PART II: Suppose $|M|s_m^* = |W|s_w^*$.

1) Suppose $|W| \leq s_m^*$. It follows that $|M|s_m^* \leq s_m^* s_w^*$, i.e., $|M| \leq s_w^*$. Let g be a pairwise stable network.

Proving that $\forall m \in M,\ s_m = |W|$ and $\forall w \in W$, $s_w = |M|$ follows the same steps as the proof of 1) of Part I).

Conversely, assume that $\forall m \in M,\ s_m = |W|$ and $\forall w \in W$, $s_w = |M|$. Proving that g is pairwise stable follows the same steps as the proof of the converse implication of 1) of Part I).

2) Suppose $|W| > s_m^*$. Let g be a pairwise stable network. Suppose that 2-(a) is not true. We should prove that 2-(b) is true.

Fact 3. $A(g) \neq \emptyset$ and $B(g) \neq \emptyset$: The fact that 2-(a) is not true implies that $A(g) \neq \emptyset$ or $B(g) \neq \emptyset$. If

$B(g) \neq \emptyset$, then, one can prove, as in Part I), that $A(g) \neq \emptyset$. Similarly, if $A(g) \neq \emptyset$, with the same argument, it is easy to prove that $B(g) \neq \emptyset$. So, $A(g) \neq \emptyset$ and $B(g) \neq \emptyset$. It remains to show that for any $m_1 \in B(g)$ and $w_1 \in A(g)$, assertions (i)-(iii) are true. The proof follows exactly as in Part I).

Now suppose that 2-(b) is not true. We have to prove that 2-(a) is true. Assume by contradiction that 2-(a) is not true. Then, $A(g) \neq \emptyset$ or $B(g) \neq \emptyset$. It follows that $A(g) \neq \emptyset$ and $B(g) \neq \emptyset$. We can therefore easily establish that for any $m_1 \in B(g)$ and $w_1 \in A(g)$, assertions (i)-(iii) are true. This contradicts the assumption that 2-(b) fails. We conclude that 2-(a) holds.

We now have to prove that, if 2-(a) or 2-(b) is true, then g is pairwise stable. Suppose that 2-(a) is true. Since each individual is at his/her peak, he/she cannot improve by severing a link or by forming a new link. It follows that g is pairwise stable.

Suppose that 2-(b) is true. The proof that g is pairwise stable follows as in Part I) under the same assumption.

PART III: Suppose $|M|s_m^* > |W|s_w^*$.

1) Suppose $|M| \leq s_w^*$. Therefore, $|W|s_w^* < |M|s_m^* \leq s_w^* s_m^*$. Hence, $|W| < s_m^*$.

Let g be a pairwise stable network. Assume by contradiction that there exists a man m_0 such that $s_{m_0} < |W|$. Recalling the fact that $|W| < s_m^*$, it follows that $s_{m_0} < s_m^*$. It also follows from the fact that $s_{m_0} < |W|$ that there exists a woman w_0 not matched with m_0. Hence, $s_{w_0} < |M|$; from the assumption that $|M| \leq s_w^*$, we therefore have $s_{w_0} < s_w^*$. Given that m_0 and w_0 are matched to fewer than their optimal number of partners each, they have an incentive to form a new match, which contradicts the fact that g is a pairwise stable network. We conclude that each man m has $s_m = |W|$ partners. This means that each man is matched to all the women, which also implies that each woman is matched to all the men. So $s_w = |M|$ for any $w \in W$.

Conversely, assume that $\forall m \in M$, $s_m = |W|$ and $\forall w \in W$, $s_w = |M|$. Since $|M| < s_w^*$ and $|W| \leq s_m^*$, no individual can improve by severing a link since he/she is at the upward sloping part of his/her utility function. Also, no individual can form a link with an opposite-type individual since he/she is already matched to all the opposite-type individuals. Therefore, g is a pairwise stable network.

2) Suppose $|M| > s_w^*$. Let g be a pairwise stable network.

Suppose that 2-(a) is not true. We should therefore prove that 2-(b) is true. Recalling Fact 1, there exists a man w_0 such that $s_{w_0} < s_w^*$. This immediately implies that $A(g) \neq \emptyset$. It is also easy to prove that $B(g) \neq \emptyset$. This is because if all men had s_m^* partners, the number of links supplied by men in g would be $|M|s_m^*$, which, by assumption, exceeds the optimal number of links that women can supply.

It remains to show that for any $m_1 \in B(g)$ and $w_1 \in A(g)$, assertions (i)-(iii) are true. The proof follows exactly as in 2) of Part I).

Now suppose that 2-(b) is not true. We should therefore prove that 2-(a) holds. Recalling Fact 1, it remains to show that $A(g) = \emptyset$. Assume by contradiction that $A(g) \neq \emptyset$. It is easy to prove that $B(g) \neq \emptyset$.

It is also easy to prove that for all $m_1 \in B(g)$ and $w_1 \in A(g)$, assertions (i)-(iii) are true. This implies that 2-(b) holds, which contradicts our assumption. Therefore, 2-(a) must hold.

We now want to prove that, if 2-(a) or 2-(b) is true, then g is pairwise stable. Suppose that 2-(a) is true. A woman cannot improve by severing a link or by forming a new link since she is at her peak. A man cannot improve by severing a link since he is at the upward sloping part of his utility function. He cannot form a new link with another woman since each woman has her optimal number of partners. Therefore, g is a pairwise stable network.

Suppose that 2-(b) is true. A man in $B(g)$ alone cannot improve by severing a link since he is at the upward sloping part of his utility function. He cannot form a new link with another woman in $A(g)$ since he is already matched to all the women in $A(g)$. He cannot form a new link with another woman in $W \setminus A(g)$ since each woman in $W \setminus A(g)$ has her optimal number of partners. A man in $M \setminus B(g)$ cannot improve by severing or forming a link since he is at his peak.

Similarly, a woman in $A(g)$ alone cannot improve by severing a link since she is at the upward sloping part of her utility function. She cannot form a new link with another man in $B(g)$ given that she is already matched to all the men in $B(g)$. She cannot form a new link with another man in $M \setminus B(g)$ since each man in $M \setminus B(g)$ has his optimal number of partners. A woman in $W \setminus A(g)$ cannot improve by severing or forming a link since she has her optimal number of partners. Therefore, g is a pairwise stable network. ■

Let us illustrate Theorem 1 with the following example.

Example 1 *Let (M, W, s_m^*, s_w^*) be an economy where $s_w^* = 3$ and $s_m^* = 6$.*

PART I.

Suppose that the economy involves 6 men $m_1 - m_6$ and 14 women $w_1 - w_{14}$ so that $|M|s_m^* < |W|s_w^*$. The two networks represented by Figure 1-1 and Figure 1-2 are pairwise stable. In Figure 1-1, each of men $m_1 - m_3$ is matched to each of women $w_1 - w_6$ and each of men $m_4 - m_6$ is matched to each of women $w_7 - w_{12}$, and women w_{13} and w_{14} are unmatched. It is easy to see that this network satisfies condition 2-(a) of Part I of Theorem 1.

In Figure 1-2, m_1 is matched to each of women $w_1 - w_6$, m_2 is matched to women $w_3 - w_7$ and w_{10}, m_3 is matched to women $w_4 - w_9$, m_4 is matched to women w_1, w_2 and $w_7 - w_{10}$, m_5 is matched to women w_1, w_2, w_3 and $w_8 - w_{10}$, and m_6 is matched to women $w_{11} - w_{14}$. We note that this network satisfies condition 2-(b) of Part I of Theorem 1 where $A(g) = \{w_{11}, w_{12}, w_{13}, w_{14}\}$ and $B(g) = \{m_6\}$.

PART II.

Suppose that the economy involves 6 men $m_1 - m_6$ and 12 women $w_1 - w_{12}$ so that $|M|s_m^* = |W|s_w^*$. The two networks represented by Figure 1-3 and Figure 1-4 are pairwise stable. In Figure 1-3, each of men $m_1 - m_3$ is matched to each of women $w_1 - w_6$ and each of men $m_4 - m_6$ is matched to each of women $w_7 - w_{12}$. It is easy to see that this network satisfies condition 2-(a) of Part II of Theorem 1.

In Figure 1-2, m_1 is matched to each of women $w_1 - w_6$, m_2 is matched to women $w_3 - w_7$ and w_{10}, m_3 is matched to $w_4 - w_9$, m_4 is matched to women w_1, w_2 and $w_7 - w_{10}$, m_5 is matched to women w_1, w_2, w_3 and $w_8 - w_{10}$, and m_6 is matched to women $w_{11} - w_{12}$. We note that this network satisfies condition 2-(b) of Theorem 1 where $A(g) = \{w_{11}, w_{12}\}$ and $B(g) = \{m_6\}$.

PART III.

Suppose that the economy involves 5 men $m_1 - m_5$ and 9 women $w_1 - w_9$ so that $|M|s_m^* > |W|s_w^*$. The two networks represented by Figure 1-5 and Figure 1-6 are pairwise stable. In Figure 1-5, m_1 is matched to each of women $w_1 - w_5$ and w_8, m_2 is matched to women $w_1 - w_5$ and w_9, m_3 is matched to $w_2 - w_7$, m_4 is matched to women w_1 and $w_6 - w_9$, and m_5 is matched to women $w_6 - w_9$. We note that this network satisfies condition 2-(a) of Part III of Theorem 1.

In Figure 1-6, each of men $m_1 - m_3$ is matched to each of women $w_1 - w_6$, and men m_4 and m_5 are matched to women $w_7 - w_9$. We note that this network satisfies condition 2-(b) of Part III of Theorem 1 where $A(g) = \{w_7, w_8, w_9\}$ and $B(g) = \{m_4, m_5\}$.

In the example above, we note that the networks represented by Figures 1-2, 1-4 and 1-6 are inefficient in the sense that the number of links supplied is smaller than the maximum number of links that can be optimally supplied. In economies where the number of men is equal to that of women, of particular interest in our dynamic stochastic analysis in the sequel, we provide a sufficient condition for pairwise stable networks to have the maximum number of links that can be optimally supplied. This class of economies is a subclass where $|M|s_m^* > |W|s_w^*$.

Proposition 1 *Let (M, W, s_m^*, s_w^*) be an economy such that (i) $|M| = |W|$ and (ii) either $s_m^* \leq \sqrt{|M|}$ or $s_w^* = 1$ or both. Let g be a network. Then, g is pairwise stable if and only if $\forall m \in M$ and $\forall w \in W$, $0 \leq s_m \leq s_m^*$ and $s_w = s_w^*$.*

Proof. Recalling the proof of assertion (2) of Part III of Theorem 1, we simply have to prove that assertion 2-(b) fails. That is, we have to prove that $\forall w \in W$, $s_w = s_w^*$. By contradiction, suppose that there exists a woman w_0 with $s_{w_0} < s_w^*$. First, it should be clear that for every man m not matched with w_0, $s_m = s_m^*$. This is because, if at least one such man were matched with fewer women, that man and w_0 would improve by forming a new link, implying that g is not pairwise stable, which is a contradiction.

It then follows that the number of links coming from the men side is at least $(|M| - s_{w_0})s_m^*$, which is greater than or equal to $[|M| - (s_w^* - 1)]s_m^*$. We will show that, under our asumptions $|M| = |W|$ and ($s_m^* \leq \sqrt{|M|}$ or $s_w^* = 1$), $[|M| - (s_w^* - 1)]s_m^*$ is greater than $|W|s_w^*$, the maximum number of links that women can supply, which is a contradiction.

Remark that $[|M| - (s_w^* - 1)]s_m^* > |W|s_w^*$ is equivalent to $|M|(s_m^* - s_w^*) > s_m^*(s_w^* - 1)$ under the assumption that $|M| = |W|$.

If $s_w^* = 1$, then, $|M|(s_m^* - s_w^*) > s_m^*(s_w^* - 1)$ is equivalent to $|M|(s_m^* - s_w^*) > 0$, which is always true, yielding the sought contradiction.

Assume that $s_m^* \leq \sqrt{|M|}$.

We have: $|M|(s_m^* - s_w^*) \geq |M|$ (because $s_m^* > s_w^*$ by assumption), and $|M| \geq (s_m^*)^2$ (by the assumption that $s_m^* \leq \sqrt{|M|}$). Since $(s_m^*)^2 > s_m^*(s_w^* - 1)$, it follows that $|M|(s_m^* - s_w^*) > s_m^*(s_w^* - 1)$, yielding the sought contradiction.

We therefore conclude that $\forall w \in W$, $s_w = s_w^*$. ■

Let us illustrate Proposition 1 with the following example.

Example 2 *Consider a matching problem in which there are 10 men and 10 women. Assume that their utility functions are such that* $s_w^* = 2$ *and* $s_m^* = 3$. *The five networks represented respectively by Figures 2-1, 2-2, 2-3, 2-4 and 2-5 are pairwise stable. In fact, in each graph, each woman has 2 partners (the optimal number of partners for each woman), and each man has at most 4 partners. In the first network component configuration* $[(2,2);(5,5);(3,3)]$[11]*, all agents have 2 partners, thus this network is egalitarian; in the second network component configuration* $[(10,10)]$*, all agents have 2 partners as in Figure 2-1; in the third network component configuration* $[(7,6);(3,4)]$*, 2 men have 1 partner each, 6 men have 2 partners each, and 2 men have 3 partners each; in the fourth network component configuration* $[(5,7);(1,0);(1,0);(2,3)]$*, 6 men have 3 partners each, 1 man has 2 partners, and 2 men have no partner; in the fifth network component configuration* $[(7,10);(1,0);(1,0)]$*, 6 men have 3 partners each, 1 man has 2 partners, and 2 men have no partner. An interesting feature of the last three graphs is the uneven share of female partners among men, which reveals a sharp competition in the latter group.*

4 A Dynamic Network Formation Process

In this section, we turn to dynamics. First, we shall define a Markov process for any given matching problem as previously defined, to describe the formation and severance of links over time. Later on, given the lack of predictive power of this process, we shall resort to perturbing it in two different ways, leading to two perturbed Markov processes, studied in Sections 4.2 and 4.3, respectively.

4.1 The Unperturbed Process P^0

The unperturbed Markov process, labelled P^0, is as follows. Time is discrete. In each period, a man and a woman chosen at random with arbitrary positive probability are given the opportunity to sever or add a link based on the improvement that the resulting network offers to them relative to the current network. If they are already linked in the current network, the decision is whether to sever the link. Otherwise, the decision

[11] $[(2,2);(5,5);(3,3)]$ refers to a network component configuration with 3 components, the first containing 2 men and 2 women, the second 5 men and 5 women, and the third containing 3 men and 3 women. This notation is a simplification that abstracts from the complete network structure as represented by the graph.

is whether to form a new link. While forming a new link, each agent is allowed to sever as many of the links he/she is involved in as possible in the current network. Link severance is unilateral, while link formation is bilateral.

Let g and g' be two networks. They are said to be adjacent if g' is obtained from g by an agent severing an existing link he/she is involved in in g, and possibly forming a new link with an agent of the opposite type. More formally, g and g' are adjacent if there exist $i \in M$ and $j \in W$ such that $g' \in \{g+ij, g+ij-ik, g+ij-ik-jm, g+ij-jm, g-ij\}$.[12] Let x and y be two networks. An $(x,y)-path$ is a finite sequence of networks $(g^0, g^1, \ldots, g^k)$ such that $g^0 = x$, $g^k = y$, and for any $t \in \{0, 1, \ldots, k-1\}$, g^t and g^{t+1} are adjacent.

An improving path from x to y is a finite sequence $x = g^0, g^1, \ldots, g^k = y$ such that for any $t \in \{0, 1, \ldots, k-1\}$, the transition from g^t to g^{t+1} strictly benefits its initiator(s). More formally:

- *(i)* $g^{t+1} = g^t - ij$ for some ij such that $u_i(s_i(g^{t+1})) > u_i(s_i(g^t))$ or $u_j(s_j(g^{t+1})) > u_j(s_j(g^t))$; or

- *(ii)* $g^{t+1} \in \{g^t+ij, g^t+ij-ik, g^t+ij-ik-jm, g^t+ij-jm\}$ for some ij such that $u_i(s_i(g^{t+1})) > u_i(s_i(g^t))$ and $u_j(s_j(g^{t+1})) > u_j(s_j(g^t))$. Here, without loss of generality, due to our single-peak assumption, there is no need to allow for an agent severing more than one link when forming a new link. If an agent has at most his/her optimal number of partners, severing more than one link when forming a new link can only lower the utility of such an agent, and there is no incentive for taking such an action. For this reason, we take the probability of such an action to be zero, just as the probability of severing a link to form a new link. In general, an agent initiates an action with probability 1 if that action strictly improves his/her utility and with probability 0 if that action does not increase his/her utility.

Recurrent classes of a Markov process are those sets of states such that, if reached, the process cannot get out of them, and which do not contain a smaller set with the same property. We next characterize the recurrent classes or steady states of the unperturbed markov process P^0.

Theorem 2 *The recurrent classes of the unperturbed markov process P^0 are singletons, whose union coincides with the set of pairwise stable networks.*

Proof. Let g be a pairwise stable network. No agent can thus be part of a blocking move either by himself/herself or with another agent, implying that there is no improving path leading out of g. $\{g\}$ is therefore a recurrent class of P^0. Conversely, if g is not pairwise stable, it cannot be part of a recurrent class of P^0. First, it is clear that if g has some agents to the right of their peaks, unilateral severance of links will constitute an improving path out of g, leading to strict individual improvements that put every agent weakly to the left of their peaks. But then, if g is not pairwise stable, and since the process has each pair being chosen with positive probability as a potential encounter, it must be the case that the blocking pair will meet. Such a link

[12] We simplify notation here and write ij instead of (i, j), $g + ij$ instead of $g \cup \{(i, j)\}$, and $g - ij$ instead of $g \setminus \{(i, j)\}$, etc.

will be formed in an improving path, never to return to g. This contradicts that g is part of a recurrent class of P^0. ■

Thus, the set of long-run predictions of the unperturbed dynamics is quite large (recall the characterization in Theorem 1). We proceed by perturbing this process in the sequel. We shall define below two such perturbed processes. For simplicity in the stochastic analysis that follows, we shall assume, following Proposition 1, that there is an equal number of men and women and that the population is sufficiently large relative to the optimal numbers of partners:

$$\textbf{A1:}\ \text{(i)}\ |M| = |W|;\ \text{and (ii) either}\ s_m^* \le \sqrt{|M|}\ \text{or}\ s_w^* = 1\ \text{or both.}$$

4.2 The First Perturbed Markov Process P_1^{ε}

In this subsection, we define and analyze the first perturbed process. In each period, the revision opportunity offered at random to a male-female pair is the same as described in the process P^0. However, now agents may make decisions that do not necessarily lead to an immediate individual improvement. We describe these events in detail.

- If the two agents are linked in the current network:
  - Link severance takes place with probability 1 if it benefits either of the two agents, just as before.
  - Otherwise, while in the unperturbed process no severance of this link was taking place, now if it makes the two agents worse off, severance takes place with probability ε (note that in our model, link severance cannot make an agent indifferent). Recall that link severance is a unilateral decision, and thus it takes one "mistake" to sever such a good link: an agent making a mistake with probability $\varepsilon > 0$.
- If the two agents are not linked in the current network, the decision is whether to form a new link:
  - This link formation takes place with probability 1 if it is mutually beneficial, just as before. All other transitions did not happen in the unperturbed process, while now they will.
  - If forming the link makes one agent worse off and the other better off –one "mistake"–, it occurs with probability ε.
  - If the link formation makes the two agents worse off –two "mistakes"–, it occurs with probability ε^2.
  - If the transition makes one agent better off and the other agent, say j, indifferent, agent j may take this "neutral action" and looks at considerations other than his/her well-being. Indifference in the transition happens because, while forming a new link with i, j severs an existing link, say with agent k in the current network. Then, the resistance of this transition amounts essentially to the

perceived strength (or quality) of the severed link. Specifically, we assume that the transition occurs with probability $\varepsilon^{f(\frac{1}{s_k})}$ where the link strength f is a strictly increasing function of $\frac{1}{s_k}$ mapping into $(0, 1)$. Here, s_k is the number of partners that k has in the current network. We offer an interpretation below, at the end of the description of the process.

– If the transition makes one agent worse off and the other agent indifferent (one "mistake" and one "neutral action"), the transition occurs with probability $\varepsilon \times \varepsilon^{f(\frac{1}{s_k})} = \varepsilon^{1+f(\frac{1}{s_k})}$.

– Finally, if it makes the two agents indifferent (two "neutral actions"), meaning that while forming a new link, i and j severed links with, say h and k, respectively in the current network, it occurs with probability $\varepsilon^{f(\frac{1}{s_h})} \times \varepsilon^{f(\frac{1}{s_k})} = \varepsilon^{f(\frac{1}{s_h})+f(\frac{1}{s_k})}$.

We emphasize our assumption on the resistance of transitions involving indifferences or "neutral actions", the key transitions for our results. The function $f(\frac{1}{s_k})$ can be viewed as the perceived strength of the link that is being severed by j. If we assume for instance that each agent is endowed with 1 unit of time that he/she splits equally among all his/her partners, then it makes sense to assume that the strength of a link is inversely proportional to the number of partners.[13]

The time invested in a relationship may also be viewed as proof of commitment to the other partner.[14]

4.2.1 Resistance of a Path and Stochastic Stability

For any adjacent networks g and g', the resistance of the transition from g to g', denoted $r(g, g')$, is the weighted number of agents directly involved in the transition who do not find this change profitable; it is the exponent of ε in the corresponding transition probability. We explicitly define $r(g, g')$ in the table below, as a function of the possible frictions –"mistakes" or "neutral actions"– found in a randomly chosen pair (i, j). To read the table, note that there are only three actions that either i or j can take, some combinations of which might not be possible:

$\mathcal{A}$- Forming a new link without severing an existing link.

$\mathcal{B}$- Forming a new link while severing an existing link.[15]

$\mathcal{C}$- Severing an existing link.

Let (a_i, a_j) be the pair of actions taken by i and j, respectively. Then $(a_i, a_j) \in \{(\mathcal{A}, \mathcal{A}), (\mathcal{A}, \mathcal{B}), (\mathcal{B}, \mathcal{B}), (\mathcal{C}, \mathcal{C})\}$. A pair of actions (a_i, a_j) might make either agent better off (b), lose (l), or indifferent (i). Transition probabilities and resistances are summarized in Table 1 below.

[13] Although for simplicity, we assume that j observes s_k, slightly weaker assumptions would do, as j could evaluate the strength $f(\frac{1}{s_k})$, for instance through a noisy signal, such as the amount of time spent by the partner out of the house. We do not model incomplete information in this paper: a next step in the analysis of the model would be not to assume observability of the number of your partner's partners. For the use of stochastic stability, the agent may not be aware of the exact probability of each event happening, which is just a parameter of the overall dynamics.

[14] One could consider a related model that avoids perturbations of the basic Markov process. In it, agents' preferences are lexicographic with respect to number of links and neutral actions in that order (that is, partner's number of partners). However, we note that the models are not equivalent. Our stochastic processes always lead to Pareto-efficient networks, whereas networks obtained under lexicographic preferences are not Pareto-efficient in general, even in this subclass.

[15] Forming a new link while severing more than one link, if not utility improving, is a transition with strictly higher resistance than the one severing only one link, and hence, it can be safely ignored in the subsequent analysis.

Table 1

(a_i, a_j)	*Outcomes*	*Probability*	$r(g, g') = \log_{\varepsilon}(probability)$
$(\mathcal{A}, \mathcal{A})$	(b, b)	1	0
$(\mathcal{A}, \mathcal{A})$	(b, l)	ε	1
$(\mathcal{A}, \mathcal{A})$	(l, l)	ε^2	2
$(\mathcal{A}, \mathcal{B})$	(b, i)	$\varepsilon^{f(\frac{1}{s_k})}$	$f(\frac{1}{s_k})$
$(\mathcal{A}, \mathcal{B})$	(l, i)	$\varepsilon^{1+f(\frac{1}{s_k})}$	$1 + f(\frac{1}{s_k})$
$(\mathcal{B}, \mathcal{B})$	(i, i)	$\varepsilon^{f(\frac{1}{s_h})+f(\frac{1}{s_k})}$	$f(\frac{1}{s_h}) + f(\frac{1}{s_k})$
$(\mathcal{C}, \mathcal{C})$	(b, b)	1	0
$(\mathcal{C}, \mathcal{C})$	(b, l)	1	0
$(\mathcal{C}, \mathcal{C})$	(l, l)	ε	1

The resistance of an (x, y)-path $q = (g^0, g^1, \ldots, g^k)$ is the sum of the resistances of its transitions: $r(q) = \sum_{t=0}^{k-1} r(g^t, g^{t+1})$.

Let $Z^0 = \{g^0, g^1, \ldots, g^l\}$ be the set of absorbing states of the unperturbed process (the pairwise stable networks, in our case).[16] Consider the complete directed graph with vertex set Z^0, denoted ∇. The resistance of the edge (g^i, g^j) in ∇ is the minimum resistance over all the resistances of the $(g^i, g^j) - paths$: $r(g^i, g^j) = minimum\{r(q) \mid q \text{ is an } (g^i, g^j)\text{-path}\}$.

Let g be an absorbing state. A g-tree is a tree whose vertex set is Z^0 and such that from any vertex g' different from g, there is a unique directed path in the tree to g. The resistance of a g-tree is the sum of the resistances of the edges that compose it. The stochastic potential of g, denoted $r(g)$, is the minimum resistance over all the g-*trees*.

The set of stochastically stable networks is the set $\{g \mid r(g) \leq r(g') \text{ for all } g'\}$ (Young (1993), Kandori, Mailath and Rob (1993)). Intuitively, this set is the set of states (or networks in our case) that are visited a positive proportion of time in the long run. They are also the networks which are the easiest to transition to.

4.2.2 The Result

We shall now characterize the set of stochastically stable states (or networks) of the perturbed process P_1^{ε}. The following definitions and lemmas, of interest in their own right, are needed.

Let g be a network. We shall say that g is egalitarian if all vertices have the same degree; that is, if all individuals have the same number of partners.

Pose $I(g) = \{i \in M : s_i(g) \geq s_j(g) \ \forall j \in M\}$, i.e., the set of men who are linked to the highest number of women in the network g.

Let $J(g) = \{i \in M : s_i(g) \leq s_j(g) \ \forall j \in M\}$, i.e., the set of men who are linked to the lowest number of women in the network g.

And call $I^*(g) = \{i \in M : s_i(g) \geq s_w^*\}$, i.e., the set of men who have at least a number of partners no less than the women's optimal number.

[16] Absorbing states are those in singleton recurrent classes.

It is obvious that, if g is pairwise stable, $I(g)$, $J(g)$ and $I^*(g)$ are non-empty. Let $L(g) = \sum_{i \in I^*(g)}(s_i(g) - s_w^*)$.

The following lemma states that, under our Assumption A1, any non-egalitarian pairwise stable network (or network in which agents do not all have the same number of partners) is such that any man in $I(g)$ is matched with more than s_w^* partners, and any man in $J(g)$ is matched with less than s_w^* partners.

Lemma 1 *Assume A1, and let g be a non-egalitarian pairwise stable network. Then, $\forall(i,j) \in I(g) \times J(g)$, $s_i(g) > s_w^* > s_j(g)$ (and therefore, $s_i(g) \geq s_j(g) + 2$).*

Proof. Appealing to the characterization of pairwise stable networks in Proposition 1 and using the definition of egalitarian networks, the proof is easy and left to the reader. ∎

The following lemma describes a simple way to reach egalitarian networks travelling through pairwise stable networks from any initial pairwise stable network.

Lemma 2 *Assume A1, and let g be a pairwise stable network. Then, there exists a finite sequence of adjacent pairwise stable networks $(g^0, g^1, \ldots, g^k)$ such that $g^0 = g$, $g^k = g^{L(g)}$, and g^k is egalitarian.*

Proof. Let g be a pairwise stable network. Pose $g^0 = g$. If g is egalitarian, then $\forall i \in M \cup W$, $s_i(g) = s_w^*$. Thus $L(g) = \sum_{i \in I^*(g)}(s_i(g) - s_w^*) = 0$, implying that the sequence searched for is (g). If g is non-egalitarian, then it is obvious that $L(g) > 0$ since from Lemma 1, at least one man has more than s_w^* partners. There exists a pair of men $(i_0, j_0) \in I(g) \times J(g)$. Again by Lemma 1, since $s_{i_0}(g) \geq s_{j_0}(g) + 2$, there exists a woman k_0 such that $(i_0, k_0) \in g$ and $(j_0, k_0) \notin g$. Sever the link (i_0, k_0), and add the link (j_0, k_0); call the resulting network g^1. It is easy to check that g^1 is pairwise stable and that $L(g^1) = L(g) - 1$. Then, either g^1 is egalitarian and we are done, or not. That is, repeating the same operation $L(g) - 1$ more times induces a sequence $(g^1, \ldots, g^{L(g)})$ of pairwise stable networks. We have $L(g^{L(g)}) = L(g) - L(g) = 0$. Therefore, in the network $g^{L(g)}$, no man has more than s_w^* partners. But given that each woman has s_w^* partners in $g^{L(g)}$, that $|M| = |W|$, and that $\sum_{i \in M} s_i(g^{L(g)}) = \sum_{j \in W} s_j(g^{L(g)}) = s_w^*|W|$, it is necessarily the case that $\forall i \in M$, $s_i(g^{L(g)}) = s_w^*$. Thus $g^{L(g)}$ is pairwise stable and egalitarian. ∎

In addition, any two egalitarian pairwise stable networks are "connected" through a path where at least half of it consists of egalitarian pairwise stable networks. This is shown in the following connectivity lemma:

Lemma 3 *Assume A1, and let g and g' be two distinct egalitarian pairwise stable networks. Then, there exists a finite sequence of adjacent pairwise stable networks $(g^0, g^1, \ldots, g^{2k})$ such that $g^0 = g$, $g^{2k} = g'$, and for any t such that $0 \leq t \leq k$, g^{2t} is egalitarian.*

Proof. Let g and g' be two distinct egalitarian pairwise stable networks. Pose $g^0 = g$. Pose $g' \setminus g = \{(m, w) : (m, w) \in g' \text{ and } (m, w) \notin g\}$. Since g and g' are different, $g' \setminus g$ is non-empty. Thus, there exists a pair (m_0, w_0) such that $(m_0, w_0) \in g'$ and $(m_0, w_0) \notin g$. Since g and g' are egalitarian, this implies that there exists a man m_0' such that $(m_0', w_0) \in g$ and $(m_0', w_0) \notin g'$. (In fact, if we assumed by contradiction that the latter statement were wrong, then it would mean that for any pair $(m_0', w_0) \in g$, then $(m_0', w_0) \in g'$; and since $(m_0, w_0) \in g'$ and $(m_0, w_0) \notin g$, this would imply that w_0 has more than s_w^* in the network g', contradicting the fact that g' is egalitarian and pairwise stable.)

Then, in g, add the link (m_0, w_0) and delete the link (m_0', w_0) (this is equivalent to woman w_0 severing her link with m_0' to form a new link with m_0), and call the resulting network g^1. In g^1, m_0 and m_0' have respectively $s_w^* + 1$ and $s_w^* - 1$ partners, and each woman has s_w^* partners as in g. Thus g^1 is pairwise stable, but it is not egalitarian. Also, note that g^1 is (one step) closer to g' than $g^0 = g$ (that is, $g' \setminus g^1 \subset g' \setminus g$).

We now want to construct g^2. Let $g^1(m_0) = \{w \in W : (m_0, w) \in g^1\}$. There exists a woman $w_0' \in g^1(m_0)$ such that $w_0' \neq w_0$, $(m_0', w_0') \notin g^1$ and $(m_0, w_0') \notin g'$ (in fact, since $|g^1(m_0)| = s_w^* + 1 > 1$ and $w_0 \in g^1(m_0)$, there exists $w_0' \in g^1(m_0)$ such that $w_0' \neq w_0$; now, if by contradiction, we assume that for any such w_0', $(m_0', w_0') \in g^1$, then it will turn out that $|g^1(m_0')| = s_w^*$, which is a contradiction since we know from the last paragraph that m_0' has exactly $s_w^* - 1$ partners in g^1; finally, if by contradiction, we assume that for any such w_0', $(m_0, w_0') \in g'$, then it will turn out that $g'(m_0) = g^1(m_0)$, implying that $|g'(m_0)| = s_w^* + 1$, thereby contradicting the fact that g' is egalitarian). Therefore, sever the link (m_0, w_0'), add the link (m_0', w_0'), and call the resulting network g^2. It is easy to check that in g^2, each man and each woman have exactly s_w^* partners. Thus g^2 is egalitarian and pairwise stable.

We also note that g^2 is at least 1 step closer to g' (in fact, since $(m_0, w_0') \notin g'$, severing this link in g^1 does not take us 1 step further from g'; also, if possible, one can choose w_0' in such a way that $(m_0', w_0') \in g'$, and in that case, g^2 will be 2 steps closer to g'; if not, g^2 will be 1 step closer to g').

If $g^2 = g'$, we are done; if not, repeat the same operation as previously by replacing g^0 with g^2. That will induce g^3 and g^4, and will take us at least one step closer to g'. In general, since $|g' \setminus g|$ is finite, repeating this operation a finite number of times (at most $\left\lceil \frac{|g' \setminus g|}{2} \right\rceil$ times) induces a finite sequence of pairwise stable networks $(g^0, g^1, \ldots, g^{2k})$ that ends at $g^{2k} = g'$ and satisfying that for any t such that $0 \leq t \leq k$, g^{2t} is egalitarian. ■

We are now ready to state the main result of the subsection:

Theorem 3 *Assume A1. A network is stochastically stable in the perturbed process P_1^ε if and only if it is egalitarian and pairwise stable.*

Proof. The proof is divided in two steps, as follows:

Step 1: Let g be a non-egalitarian pairwise stable network. We shall show that g is not stochastically stable. It suffices to show that there exists a network g' such that $r(g') < r(g)$.

Call $T(g)$ the g-tree on which the calculation of $r(g)$ is based. There exists a pair of men $(i_0, j_0) \in I(g) \times J(g)$. Since from Lemma 1, $s_{i_0}(g) \geq s_{j_0}(g) + 2$, there exists a woman k_0 such that $(i_0, k_0) \in g$ and $(j_0, k_0) \notin g$. Sever the link (i_0, k_0), and add the link (j_0, k_0), and call the resulting network g^1.

Consider now the tree $T(g)$. Let $S(g^1, T(g))$ be the successor of g^1 in the tree. Now, in $T(g)$, delete the edge $(g^1, S(g^1, T(g)))$ that leads away from g^1 and add the edge (g, g^1). This results in a g^1-tree that we denote by $T(g^1)$.

Since $T(g^1)$ is not necessarily optimal for g^1, we have $r(g^1) \leq r(g) - r(g^1, S(g^1, T(g))) + r(g, g^1)$. Because $\forall i \in I(g^1)$, $s_i(g) \leq s_{i_0}(g)$, we have $r(g^1, S(g^1, T(g))) \geq f(\frac{1}{s_{i_0}(g)}) = r(g, g^1)$. This is because the cheapest way of getting away from g^1 (which is pairwise stable) is for a pair of a man and a woman to undertake an action that benefits one of them and leaves the other indifferent; such an action is taken with probability at least equal to $\varepsilon^{f(\frac{1}{s_{i_0}(g)})}$. This implies that $r(g^1) \leq r(g)$.

If g^1 is egalitarian, then $r(g^1, S(g^1, T(g))) = f(\frac{1}{s_w^*}) > r(g, g^1)$, implying $r(g^1) < r(g)$. If g^1 is non-egalitarian, repeat the same operation $L(g) - 1$ more times. From Lemma 2, that will induce a sequence of pairwise stable networks $(g^1, \ldots, g^{L(g)})$ where $g^{L(g)}$ is an egalitarian network. The induced sequence of g^ℓ-trees, $1 \leq \ell \leq L(g)$, $(T(g^1), \ldots, T(g^{L(g)}))$ will be such that for any $\ell \in \{2, \ldots, L(g)\}$, $r(g^\ell) \leq r(g^{\ell-1})$ with $r(g^{L(g)}) < r(g^{L(g)-1})$. This obviously implies $r(g^{L(g)}) < r(g)$, and therefore, g is not stochastically stable.

Recall that in any perturbed finite Markov process the set of stochastically stable states is always non-empty. Step 1 has therefore established that the set of stochastically stable networks of the perturbed process P_1^ε is a non-empty subset of the set of egalitarian pairwise stable networks.

Step 2: We shall next show that the set of stochastically stable networks of P_1^ε coincides with the set of egalitarian pairwise stable networks. It suffices to show that all egalitarian pairwise stable networks have the same stochastic potential.

Let g and g' be any two egalitarian pairwise stable networks, and $r(g)$ and $r(g')$ their respective stochastic potentials. Call $T(g)$ the g-tree on which the calculation of $r(g)$ is based. From Lemma 3, we know that there exists a finite sequence of pairwise stable networks $(g^0, g^1, \ldots, g^{2k})$ such that $g^0 = g$, $g^{2k} = g'$, and for any t such that $0 \leq t \leq k$, g^{2t} is egalitarian.

Construct g^1 from g as in the proof of Lemma 3, and consider the g-tree $T(g)$. In it, delete the edge $(g^1, S(g^1, T(g)))$ that leads away from g^1 and add the edge (g, g^1). This results in a g^1-tree that we denote by $T(g^1)$. Note that $r(g^1, S(g^1, T(g))) \geq f(\frac{1}{s_w^*+1})$ and $r(g, g^1) = f(\frac{1}{s_w^*})$.

Next, construct g^2 from g^1 as in the proof of Lemma 3, and consider the g^1-tree $T(g^1)$. In it, delete the edge $(g^2, S(g^2, T(g^1)))$ and add the edge (g^1, g^2). This results in a g^2-tree that we denote by $T(g^2)$. We have $r(g^2, S(g^2, T(g^1))) = f(\frac{1}{s_w^*})$ and $r(g^1, g^2) = f(\frac{1}{s_w^*+1})$.

Therefore, noting that $T(g^2)$ is not necessarily optimal as a g^2-tree, we have that $r(g^2) \leq r(g) - r(g^1, S(g^1, T(g))) + r(g, g^1) - r(g^2, S(g^2, T(g^1))) + r(g^1, g^2) = r(g) - r(g^1, S(g^1, T(g))) + f(\frac{1}{s_w^*+1}) \leq r(g)$ since $r(g^1, S(g^1, T(g))) \geq f(\frac{1}{s_w^*+1})$. This establishes that $r(g^2) \leq r(g)$, and by symmetry, going back from g^2 to g, that $r(g) \leq r(g^2)$. Therefore, $r(g) = r(g^2)$.

If $g' = g^2$, then we have shown that $r(g') = r(g)$. If $g' \neq g^2$, repeat the same exercise as previously, constructing g^ℓ from $g^{\ell-1}$ as in Lemma 3, until g' is obtained. This induces a sequence of $g^t - trees$

$(T(g), T(g^1), T(g^2), T(g^3), \ldots, T(g^{2k}) = T(g'))$ satisfying that for any t such that $1 \leq t \leq k$, $r(g^{2t}) \leq r(g^{2(t-1)})$. This implies $r(g') \leq r(g)$. By symmetry, going back in the opposite direction, we also have $r(g) \leq r(g')$, thus implying $r(g) = r(g')$, which completes the proof. ■

The logic behind the proof of Theorem 3, as well as Theorem 4 below, has no connection with that of a model where lexicographic preferences are assumed, which is why these proofs are somewhat involved. Interestingly, our stochastic process implies that if a network that is not efficient is reached (such a network might be the outcome of a model with lexicographic preferences), there exists a path leading out of that network and towards a Pareto-efficient outcome. The interested reader may find illustrations of the workings of these theorems in Pongou and Serrano (2009, 2013), which provide examples to show how networks that are not stochastically stable transition into stochastically stable ones.

4.3 The Second Perturbed Process P_2^ε

In this subsection, we define and analyze the second perturbed process. This process is defined as the first perturbed process, the only difference being the definition of the probability of a "neutral action", an action that leaves an agent indifferent. Recall that that probability was based on the strength of the link to be broken to form the new link. Now, the *perceived* strength or quality of such a link will be inversely proportional to the amount of time invested in it. We describe next more formally the only change in assumptions with respect to the previous perturbed process:

- A person who is indifferent in a particular transition, and in it, breaks an existing link with another person who has s_k partners in order to form a new link looks at the *perceived* strength of the link he/she severs. That strength $f(s_k)$ is strictly increasing in s_k and strictly bounded between 0 and 1.

We offer some interpretations of this process. First, in a fidelity economy, this process might correspond to a situation in which an agent who invests too much time in a relationship is perceived as weak or dominated in that relationship.[17] Second, the time invested by agent k in a relationship with i might be (correctly or not) sending a signal regarding the quality of i as a partner to other agents (other agents might wrongly or rightly think that i should be of high quality for k to dedicate his/her time to her/him). In this light, an individual who has more partners might be perceived as being of higher quality, allowing him/her to attract even more partners. If the perception of higher quality related to the number of partners is wrong, then breaking a current link to form a new link effectively leaves the initiator of such an action indifferent. If this perception is right, then our second stochastic process can be viewed as being utility-driven. However, given that an agent cannot be sure of whether a potential partner who has a greater number of links is of higher quality prior to

[17] For a possible justification of this interpretation in a sexual economy characterized by male domination, see Tertilt (2005) and further evidence from anthropologists (Pat Caldwell (1976), John C. Caldwell (1976), John C. Caldwell, Pat Caldwell and Orubuloye (1992), Quale (1992)). However, as noted by Pongou and Serrano (2013), the anthropological literature only offers a post-fact rationalization of male domination, whereas the assumption underlying our neutral actions is more general given its gender neutrality: a partner is more likely to be dumped if he/she is perceived as weak, regardless of gender.

experiencing the new relationship, breaking a current link to form a new one with a higher-connected agent, *a priori*, cannot be viewed as a utility-improving action, which is why we assume that such an action is not taken with probability 1.

4.3.1 Resistance of a Path

All the definitions of resistance provided earlier apply to this section as well. For completeness, for each adjacent transition in the perturbed process P_2^ε, its probability and resistance are summarized in Table 2 below. It uses the same notation employed in Table 1:

Table 2

(a_i, a_j)	*Outcomes*	*Probability*	$r(g, g') = \log_\varepsilon(\mathit{probability})$
$(\mathcal{A}, \mathcal{A})$	(b, b)	1	0
$(\mathcal{A}, \mathcal{A})$	(b, l)	ε	1
$(\mathcal{A}, \mathcal{A})$	(l, l)	ε^2	2
$(\mathcal{A}, \mathcal{B})$	(b, i)	$\varepsilon^{f(s_k)}$	$f(s_k)$
$(\mathcal{A}, \mathcal{B})$	(l, i)	$\varepsilon^{1+f(s_k)}$	$1 + f(s_k)$
$(\mathcal{B}, \mathcal{B})$	(i, i)	$\varepsilon^{f(s_h)+f(s_k)}$	$f(s_h) + f(s_k)$
$(\mathcal{C}, \mathcal{C})$	(b, b)	1	0
$(\mathcal{C}, \mathcal{C})$	(b, l)	1	0
$(\mathcal{C}, \mathcal{C})$	(l, l)	ε	1

4.3.2 The Result

We shall now characterize the set of stochastically stable states of the perturbed process P_2^ε. The following definition is needed.

Let g be a network. We say that g is anti-egalitarian if $\left\lfloor \frac{s_w^*}{s_m^*}|M| \right\rfloor$ men are matched to s_m^* women each, at most one man is matched to the remaining women (if there is such a remainder), and all other men have no partner.

To understand this definition, the idea is that all women are matched to a set of men that is as small as possible; hence the name "anti-egalitarian." This resembles a *one-sided thin economy* with a small number of active sellers and many buyers. Thus, if $\frac{s_w^*}{s_m^*}|M|$ happens to be an integer, each of those men is matched to s_m^* women and the rest of men are unmatched. Note that if $\frac{s_w^*}{s_m^*}|M|$ is not an integer, one can assign the remaining women to only one man and have a pairwise stable network. This is because, if one calls K the integer part of that fraction, the total number of links from the set of men not matched to their optimal number must be less than s_m^*: otherwise, the number of links coming from the men side would be at least $Ks_m^* + s_m^*$, but this number is strictly greater than $s_w^*|M|$, the number of links coming from the women side, and both numbers must always be equal.

Equipped with this definition, we state our next result:

Theorem 4 *Assume A1. A network is stochastically stable in the perturbed process P_2^ε if and only if it is anti-egalitarian and pairwise stable.*

Proof. The proof is again organized in two steps, as follows:

Step 1: Let g be a pairwise stable network that is not anti-egalitarian. We shall show that g is not stochastically stable. It suffices to show that there exists a network g' such that $r(g') < r(g)$.

Consider $T(g)$, the g-tree on which the calculation of $r(g)$ is based. We claim that, if g^{λ} and $g^{\lambda+1}$ are two pairwise stable networks such that for some m, m', w, $g^{\lambda} \setminus g^{\lambda+1} = \{(m, w)\}$ and $g^{\lambda+1} \setminus g^{\lambda} = \{(m', w)\}$, the underlying transition does not involve non-pairwise stable networks: if it did, at least one agent directly involved in it would decrease his or her utility, which implies that the resistance of such a transition would exceed 1, whereas the resistance of the direct transition between the two (being adjacent) is strictly less than 1. A simple induction argument shows that this is still true even if two pairwise stable networks are not adjacent (by constructing a path going from one to the other consisting of direct transitions between pairs of adjacent networks).

Therefore, in any transition described in $T(g)$, only pairwise stable networks are visited. By Proposition 1, we know that each pairwise stable network contains exactly the same number of links, i.e., $s_w^*|W|$. It follows that each transition described in the tree involves a woman w who severs a link with a man m and replaces it with another link with man m'. Specifically, the pair (m', w) is offered the opportunity to revise their situation, and as a result, woman w severs (m, w) and gets matched with m'.

But then, in describing the transition between any two pairwise networks in $T(g)$, one can, without loss of generality, list the transitions that are required going through each individual woman. That is, starting with the woman with the lowest index who has a different set of men to which she is matched in the two networks, one can describe the required severance/creation of links that takes her from her configuration of men in the original network to the one in the final network, and one can proceed like these with each such woman until the full transition is complete.

Consider then the network g, and recall it is not anti-egalitarian. We propose the following algorithm. Without loss of generality, label the men so that $s_{m_1}(g) \geq s_{m_2}(g) \geq \ldots \geq s_{m_{|M|}}(g)$. Let m be the lowest index such that $s_m(g) < s_m^*$. If there exists w who is matched in g to $m' > m$, sever the link (m', w) and replace it with (m, w). Call the resulting network g^1. We can have two cases. Either g^1 is anti-egalitarian, or not. If it is, let $g' = g^1$. If not, repeat the same step. Note how this algorithm always ends after a finite number of steps, say k, in a network $g' = g^k$ that is anti-egalitarian.

Consider the g-tree $T(g)$, and without loss of generality (as the first paragraphs of the proof showed), suppose that the transition $g' = g^k \to g^{k-1} \to \ldots \to g^1 \to g^0 = g$ constitutes a path of directed links in $T(g)$. Change the direction of this path and consider the transition $g = g^0 \to g^1 \to \ldots \to g^{k-1} \to g^k = g'$. It is obvious that the rest of edges of $T(g)$, along with these new edges (in which the only change introduced is the direction change of previous links in $T(g)$), constitute a g'-tree, which we call $T(g')$.

We claim that $r(g') < r(g)$. Indeed, $r(g')$ is no greater than the resistance of $T(g')$, which is equal to $r(g) + \sum_{\alpha=0}^{k-1}[r(g^{\alpha}, g^{\alpha+1}) - r(g^{\alpha+1}, g^{\alpha})]$. And note that, by construction of the algorithm described, each bracketed term is negative. Indeed, in the transition $g^\alpha \to g^{\alpha+1}$, let m' be the man who loses a link in favor of man m. We know that $s_{m'}(g^\alpha) < s_m(g^{\alpha+1})$, and therefore, $r(g^\alpha, g^{\alpha+1}) = f(s_{m'}(g^\alpha)) < f(s_m(g^{\alpha+1})) = r(g^{\alpha+1}, g^\alpha)$.

We have therefore established that, if g is pairwise stable but it is not anti-egalitarian, it is not stochastically stable in the perturbed process P_2^ε. Given that the set of stochastically stable networks is non-empty, we just proved that this set is a non-empty subset of the set of pairwise stable and anti-egalitarian networks.

Step 2: We shall now prove that the set of stochastically stable networks of P_2^ε coincides with the set of pairwise stable and anti-egalitarian networks. It suffices to prove that all of them have the same stochastic potential.

Let g and g' be any two such networks. Assume for simplicity that, in each of them, exactly $\frac{s_w^*}{s_m^*}|M|$ men are matched with s_m^* each. Obviously, this must hold for both g and g'.[18]

It is easy to see that there must exist $m, m' \in M, m \neq m'$ and $w, w' \in W, w \neq w'$ such that $(m, w) \in g \setminus g'$ and $(m', w') \in g' \setminus g$. We propose the following algorithm that transforms g into g'. For each such pair of links, we describe the following steps:

- First, woman w severs her link to man m and gets matched to man m_0, where $s_{m_0}(g) = 0$–we know such a man exists in g.
- Second, woman w' severs her link to man m' and gets matched to man m.
- And third, woman w severs her link to man m_0 and gets matched to man m'.

And to go back, travel the same steps in reverse.

Consider now an optimal g'-tree, and call it $T(g')$. In it, focus on the collection of directed edges connecting g to g'. By arguments similar to those at the beginning of Step 1 of this proof, one can argue that the transition outlined in the previous algorithm must be part of any optimal tree. (We know that transitions in optimal trees do not go through non-pairwise stable networks. In addition, a resistance of $f(s_m^*)$ must be paid every time a link with a man matched to his optimal number is broken, and aside from that, a resistance of $f(1)$ that comes from breaking a link with a man who was unmatched in g and remains unmatched in g' is the smallest possible positive resistance in this perturbed process.)

Thus, without loss of generality, let the directed path from g to g' in $T(g')$ be the set of transitions outlined. Now, change the direction of the edges in this path, and let that be the only change introduced to the directed edges of $T(g')$. Observe that the result is a g-tree, which we call $T(g)$.

We will now argue that the stochastic potentials of g and g' are the same:

[18] If, instead, the number $\frac{s_w^*}{s_m^*}|M|$ is not an integer, and one man is matched to the remaining women, the argument is the same, but the notation is slightly more complicated. Again, in this case, both g and g' have the same structure of having only one man matched to the remaining women.

$r(g) = r(g') + \sum_{\beta=0}^{k-1}[r(g^\beta, g^{\beta+1}) - r(g^{\beta+1}, g^\beta)] = r(g')$ because $\sum_{\beta=0}^{k-1}[r(g^\beta, g^{\beta+1}) - r(g^{\beta+1}, g^\beta)] = 0$. This can be easily established by induction on the number of links that are different between g and g'.

Indeed, suppose that g and g' differ in the smallest possible number of links, which is two, i.e., there exist $m \neq m'$ and $w \neq w'$ such that $g \setminus g' = \{(m, w)\}$ and $g' \setminus g = \{(m', w')\}$. Consider the transition $g \to g'$ in $T(g')$. By our previous arguments, such a transition is as follows:

- First, woman w severs her link to man m and gets matched to man m_0, where $s_{m_0}(g) = 0$–we know such a man exists in g; the resistance of this step is $f(s_m^*)$.

- Second, woman w' severs her link to man m' and gets matched to man m; again, the resistance of this step is $f(s_m^*)$.

- And third, woman w severs her link to man m_0 and gets matched to man m'; the resistance of this step being $f(1)$.

The resistance of the whole transition is thus $2f(s_m^*) + f(1)$. But notice that travelling the same steps backwards takes us back from g' to g, with exactly the same resistance.

If g and g' differ by more links (note this must always be an even number), we use the fact that the path going from g to g' and the same path travelled in the opposite direction are "mirror images" of one another. Thus, since the cheapeast transition must always involve establishing links with unmatched men –like m_0 in the previous paragraph– (because $f(1)$ is the smallest resistance to be added to the $f(s_m^*)$ terms, which must be always there), a replication of the argument detailed in the previous paragraph establishes that the total resistance of travelling from g to g' is exactly the same as the one travelling backwards on the same path. This completes the proof. ■

We illustrate the two stochastic processes by the economy described in Example 2. This economy involves 10 men and 10 women, and $s_m^* = 3$ and $s_w^* = 2$. We have seen that the networks described by Figures 2-1, 2-2, 2-3, 2-4 and 2-5 are all pairwise stable. However, of these networks, only the egalitarian pairwise stable network described by Figures 2-1 and 2-2 are stochastically stable under the process P_1^ε, while only the anti-egalitarian pairwise stable networks described by Figures 2-3 and 2-5 are stochastically stable under the process P_2^ε. An interesting feature of this example is that, although the stochastically stable networks (under each process) have the same distribution of links among agents, they do not have the same configuration. As we will see in Section 5.2, this difference in the configuration of long-run equilibrium networks has implications for how the spread of a random contagion shock may not affect identical economies in the same way, as a different equilibrium may realize in each of these economies.

Although technically challenging, it would be interesting to analyze stochastically stable networks in cases where $|M| > |W|$ and $|M| < |W|$. We conjecture that our conclusions regarding the allocation of links should not "qualitatively" differ in those cases. This is very likely to be the case for the second process in particular, as the proof of Theorem 4 makes little use of the assumption that $|M| = |W|$. A potential challenge in establishing the more general results would be to determine whether the inefficient networks that are pairwise stable, and hence that are recurrent classes, are stochastically stable under each of our two perturbed processes. It seems that these networks are unlikely to pass the stochastic stability test, using our perturbations based on neutral actions. To see this, suppose that g is an inefficient pairwise stable network, and that a man m in $B(g)$ and a woman w in $A(g)$ are matched (from Theorem 1, $A(g)$ and $B(g)$ represent the sets of women and men who do not have their optimal number of partners in g). By a neutral action, m could sever that link and link with another woman outside of the set $A(g)$, this woman possibly also taking a neutral action; but then in the next transition, we would see m and w matching again at zero resistance, that is, we would not return to the original network g. It follows that inefficient pairwise stable networks are unlikely to be stochastically stable under either of our two processes. While these are only conjectures and the proofs need to be worked out, strong intuition seems to support our assertion.

5 Applications and Discussion

This section covers the implications of our model and suggests several interpretations of the theoretical findings for markets where agents derive utility only from the number of partners they have–the volume of trade–. In particular, we argue that our model explains stylized facts related to fragmentation, herding, concentration, and contagion asymmetry observed in several two-sided economies. We also discuss our contribution in relation to the literature on two-sided matching and networks.

5.1 Fragmentation, Herding, and Concentration in Two-Sided Economies

Buyer-Seller Markets. Our model might be applied to illustrate the patterns of trade between sellers (M) and buyers (W) in several markets. In general, the optimal number of buyers (s^*_m) for each seller is greater than the optimal number of sellers (s^*_w) a buyer can purchase from. Despite having this feature in common, markets greatly differ in how they allocate commercial relationships between sellers and buyers. Indeed, certain markets, such as the international market for manufactured foods are highly fragmented, being served by many small sellers, whereas other markets, such as the market for military goods, are highly concentrated, involving only a small number of sellers. Our two stochastic processes can rationalize these different trade patterns.

To see how, consider, for instance, an international market in which richer countries sell manufactured goods to poorer countries. Our first stochastic process may be viewed as follows: forced to replace one trading partner with another, a country stays away from countries with already many other trading partners, in order to avoid long queues in the delivery of goods. This process therefore depicts a very competitive market, which implies that countries on each side have the same number of trading partners. It sheds light on the functioning and outcomes of the international markets for "nonstrategic" goods such as food and clothing, which involve a large number of suppliers and purchasers.

Our second stochastic process leads to a concentrated market: all the poor countries are matched to only a few number of rich countries. This process might be describing the behavior of agents in the international market for certain "strategic" goods such as military weapons. If poor "neighboring" countries compete for military leadership in their region, they will all purchase weapons and other military goods from the same suppliers in order to make sure that none of the countries has a superior war technology. This behavior is illustrated by a market involving three rich countries and three poor countries that are neighbors. Each supplier desires to sell weapons to all the poor countries, but each of the latter countries can purchase weapons from only one country. It is possible that the initial matching is the one-to-one matching in which country m_i sells weapons to country w_i, $i = 1, 2, 3$. However, if weapons are differentiated products, countries w_2 and w_3 might become fearful of supplier m_1 offering a different or better product to country w_1 (even if this perception is wrong), leading them to break their respective commercial relationships with m_2 and m_3 in order to match with m_1. This will ensure that all the poor countries possess the same war technology in the long run.

It follows that our second stochastic process might, in general, be viewed as offering an alternative model of "herd externality" (Banerjee (1992)) or "informational cascade" (Bikhchandani, Hirshleifer, and Welch (1992)) among buyers, in that, in choosing whom to purchase from, they are influenced by the choices of other buyers. Herd behavior characterizes several choices in real life, including the choice of a restaurant or of a school. Our analysis implies that herding leads to a concentrated economy, which in turn implies that the prediction of the second stochastic process, in terms of the allocation of links, partially coincides with that of the static process in the case where the total supply of links is equal for both sides of the economy (that is, $|M|s_m^* = |W|s_w^*$). However, the two models differ in that, in the static model, the number of sellers is assumed to be "exogenously" smaller than the number of buyers, whereas in the second stochastic process, the imbalance between the "active" number of sellers and buyers arises as a long-run outcome. The analysis implies that the second stochastic process embeds two interesting features. It first selects a few lucky sellers from a larger pool of sellers, and second, it induces a sharp competition among these selected sellers who generally end up having an equal share of buyers.

Although we have emphasized the application of our stochastic processes to buyer-seller markets, their interpretations are also valid for other two-sided markets including labor and academic markets. For instance, under the first process, an employer might perceive an employee who works for other competing employers as lacking commitment, and an individual who works for a firm that has many other workers might believe his career opportunities within the firm to be limited due to a perception of high competition. Under the second process, an employer who has many workers might be perceived by other workers as offering better working conditions, whereas an individual who is solicited by many firms might be perceived to be of higher quality. Under the first process, the economy will be served by many small employers matched to the same number of workers, whereas under the second process, the economy will be served by a small number of big employers.

Importantly, we remark that the interpretations of our two stochastic processes offered so far emphasize the perception of the number of partners as a signal of commitment or quality. This perception might be wrong, which is why, under these interpretations, these processes cannot be viewed as being utility-driven. However, there exists an alternative interpretation of these processes that incorporates utility considerations. Indeed, even if the quality signal sent by the number of partners that an agent has is incorrect at the initial stage of network formation (because all agents on each side of the market are initially identical in terms of the characteristics that the agents on the other side value), it might become true over time. For instance, there are many contexts where a seller's quality increases with the number of buyers he has had, as buyers generally provide feedback that help the seller improve the quality of his products. Therefore, it is possible that sellers are initially identical, and that a pairwise stable matching between sellers and buyers forms in the initial period, with some sellers having more buyers than others. Over time, sellers who initially had more buyers will have a better reputation than those who had less, even attracting more buyers up to their optimal number. The same logic applies to labor and academic markets. For instance, an employer's managerial experience and skills might increase with the number of employees he has had, and in a faculty-student market, a faculty who has had more students might over time become more skilled at advising and might have a better experience with finding better jobs for his students than his colleague who had no students, even if both were initially identical. In all of these cases, the prediction of the model is that of pairwise stable anti-egalitarian networks in the long run, in which only a small number of employers or faculty members are matched with agents on the opposite side of the economy.

Dating and union patterns across cultures. Our analysis may also be used to shed light on dating and union formation patterns in certain societies. Imagine that women's optimal number of partners is 1 (this corresponds to the official constraint on marriage for women in almost all societies). Then, in the first process, the model predicts a situation of *serial monogamy*. Theorem 3 shows for this case that only monogamous networks are stable in the long run. This notion of stability, however, does not imply that if the process reaches a monogamous network, it will stay there, since people might still make mistakes or be tempted by other potential partners. Indeed, if a woman moves from her only partner to another one, creating a non-monogamous network, the latter network will transition to another monogamous network which is not necessarily the initial one, and so on. Serial monogamy, known to be more prevalent in Western societies, is associated with high divorce rates (e.g., Schoen and Standish (2001) and Goldstein (1999) document that the divorce rate in the United States is above 40%). In contrast, under the second process, the prediction of the model is *polygyny* (polygamy involving several women matched to one man), and then divorce rates may be low. Consider the following example. There are 3 men and 3 women, and $s_w^* = 1$ and $s_m^* = 3$. Theorem 4 tells us that the only stochastically stable network (up to permutations) is the one in which the first man is matched to all three women. Assume that the process reaches that network. If a woman leaves the first man to match with another man, then considering that networks evolve following the path of least resistance, it is easy to see that that woman will return to the first man (so, there will be reconciliation and no divorce). The model may be suggesting union formation patterns in regions where polygyny coexists with low divorce rates.

Clearly, the second process better describes male-dominant societies. In such societies, as predicted by the model, men invest less time in their relationships with their wives.[19] It is also possible, as noted earlier, that the number of partners that an individual has sends a positive signal about his/her quality, which helps to attract other partners. Indeed, quoting George Bernard Shaw, Becker (1974) writes that "the maternal instinct leads a woman to prefer a tenth share in a first rate man to the exclusive possession of a third rate." The interpretations that the time invested in a partner either signals his/her dominant role or his/her quality are both consistent with our second process.

It is also important to note that monogamous and polygynous networks are only the official marital networks, as in general, women are officially allowed to marry only one husband at a time. In reality, data show that men and women cheat on their partners. When cheating is allowed, one wonders whether the networks that result share similar properties with the official marital networks, which are either monogamous or polygynous. Our findings suggest that they do. Indeed, when cheating is taken into account, according to Theorem 3, the sexual network that forms in a society where only monogamy is legal is a *union* (in the mathematical sense of the word) of monogamous networks. Similarly, it follows from Theorem 4 that the sexual network that forms in a society where polygyny is legal is a *union* of polygynous networks. In other words, the rationale that governs the formation of official marital relationships is the same that governs the formation of unofficial relationships in monogamous and polygynous cultures.

5.2 Contagion Asymmetry across Economies and Agents Types

In this subsection, we apply our findings to show how the concentration of a random *unanticipated* information shock is likely to vary across structurally identical economies depending on the "realization" of stochastically stable networks. We also answer the question of which side of the economy is more affected by the spread of a shock.

We study the concentration of a shock using the notion of the contagion potential of a network introduced in Pongou (2010). We first recall this notion. Let g be a network that has k components $g_1, \ldots, g_k$. Pose $I_k = \{1, \ldots, k\}$. To simplify notation, we write $N(g_i) = N_i$, $M(g_i) = M_i$, $W(g_i) = W_i$, and $|N_i| = n_i$ for $i \in I_k$. We associate each component g_i with the number n_i and its bipartite component vector $(|M_i|, |W_i|)$, and g with the vector $[(n_i)]_{i \in I_k}$ and its bipartite vector $[(|M_i|, |W_i|)]_{i \in I_k}$. Also, if g_i is an isolated component (a component consisting of one agent), its associated vector is either $(1, 0)$ or $(0, 1)$.

Assume that an agent is drawn at random to receive a piece of information that he/she communicates

[19] For example, based on a survey conducted in Nigeria, John C. Caldwell (1976) wrote that "fewer than one-third of wives normally eat with their husbands or seat together on occasion", which markedly contrasts with what is observed in Western societies.

to his/her partners, who in turn communicate it to their other partners, and so on. An agent who is not matched does not spread information. Assuming that agents are drawn with equal probability, $\frac{1}{|N|}$, to receive information, Pongou (2010) defines the communication or contagion potential of a network g, denoted $\mathcal{P}(g)$, as the expected proportion of agents who will receive the information. The difference across types in contagion potential, denoted $\mathcal{F}(g)$, is the difference in the expected proportion of men and women who will receive the information. These notions are formalized below.

Definition 1 *Let g be a k-component network with the corresponding component vector $[(n_i)]_{i\in I_k}$.*

(1) The communication or contagion potential of g is defined as

$$\mathcal{P}(g) = \frac{1}{n^2}\sum_{i\in I_k} n_i^2.$$

(2) If g is a bipartite graph with the corresponding component vector $[(|M_i|, |W_i|)]_{i\in I_k}$ and $|M| = |W|$, the difference across types (or male-female difference) in the contagion potential of g is defined as

$$\mathcal{F}(g) = \frac{2}{n^2}\sum_{i\in I_k}(|M_i|^2 - |W_i|^2).$$

These contagion indices assume that the transmission probability is 1. This assumption is correct if contagion means mechanically communicating a received message or a new idea. However, when the transmission probability per interaction is smaller than 1, our assumption is motivated by the fact that we are studying transmission in "long-run equilibrium" or "long-run stable" networks, which implies that contagion-prone interactions are repeated over time, causing the transmission probability to approach 1. In fact, if we assume that the transmission probability per interaction is $\lambda < 1$, and that transmission is independent across interactions, then the transmission probability after k interactions is $1 - (1 - \lambda)^k$, which effectively goes to 1 as k goes to infinity. This logic is justified in our model since our comparative statics is on stochastically stable networks. We also note that our model generalizes to situations in which agents communicate information to their partners' partners directly, such as in a classroom where students interact among themselves in addition to interacting with the instructor.

In general, a higher optimal number of partners for men than for women does not necessarily cause the latter to be more affected by the spread of a random information shock than the former, when one considers all pairwise stable networks. But we next show that, in the networks that are visited a positive proportion of time in the long run (under our perturbed processes P_1^ε and P_2^ε), information never concentrates more among men than among women.

Theorem 5 *Assume A1.*

(1) For any stochastically stable network g of the perturbed process P_1^ε, $\mathcal{F}(g) = 0$.

(2) For any stochastically stable network g of the perturbed process P_2^ε, $\mathcal{F}(g) < 0$.

Proof. Assume A1.

(1) The proof follows from the fact that in any egalitarian pairwise stable network g, there is an equal number of men and women in each component of g, from which it follows that $\mathcal{F}(g) = 0$.

(2) First remark that in any anti-egalitarian pairwise stable network g, the number of women exceeds the number of men for all non-isolated components, with strict inequality for some of them. Let us now show that it follows that $\mathcal{F}(g) < 0$. Let $[(|M_i|, |W_i|)]_{i\in I_k}$ be the bipartite component vector of g, of which the first ℓ components are non-isolated and the remaining $k - \ell$ are isolated (men). It obviously follows that $\sum_{i\in I_k} |M_i| = \sum_{i\in I_\ell} |M_i| + k - \ell$ and $\sum_{i\in I_k} |M_i| = \sum_{i\in I_\ell} |W_i|$ (given that no woman is isolated), which in turn implies $\sum_{i\in I_\ell}(|M_i| - |W_i|) = -(k - \ell) < 0$. Remark that each non-isolated component vector $(|M_i|, |W_i|)$ is such that $|M_i| + |W_i| = n_i \geq 2$ since it contains at least one man and one woman. Hence, we have the following:

$$\begin{aligned}
\mathcal{F}(g) &= \frac{2}{n^2}\sum_{i\in I_k}(|M_i|^2 - |W_i|^2) \\
&= \frac{2}{n^2}\{\sum_{i\in I_\ell}(|M_i|^2 - |W_i|^2) + \sum_{\ell+1\leq i\leq k}(|M_i|^2 - |W_i|^2) \\
&= \frac{2}{n^2}\{\sum_{i\in I_\ell}(|M_i| - |W_i|)(|M_i| + |W_i|) + \sum_{\ell+1\leq i\leq k}(1^2 - 0^2) \\
&= \frac{2}{n^2}\{\sum_{i\in I_\ell}(|M_i| - |W_i|)n_i + k - \ell \\
&\leq \frac{2}{n^2}\{2\sum_{i\in I_\ell}(|M_i| - |W_i|) + k - \ell\} \\
&< 0.
\end{aligned}$$

■

Our analysis implies that structurally identical economies might be affected differently by a random contagion shock. To illustrate, consider Example 2 and the pairwise stable networks represented by Figures 2-1, 2-2, 2-3, 2-4 and 2-5. Call them g_1, g_2, g_3, g_4, and g_5, respectively. Under the first stochastic process, g_1 and g_2 may realize in two identical economies. These networks have the same link distribution, but have different configurations. Following the spread of a random contagion shock, 38% of agents will be affected in g_1 and 100% of agents in g_2 ($\mathcal{P}(g_1) = 0.38$ and $\mathcal{P}(g_2) = 1$). Similarly, under the second stochastic process, g_4 and g_5 may realize in two identical economies. Following the spread of a random contagion shock, 42.75% of agents will be affected in g_4 and 72.75% of agents in g_5 ($\mathcal{P}(g_4) = 0.4275$ and $\mathcal{P}(g_5) = 0.7275$). These calculations show that network configuration has an independent effect on the diffusion of information. An important implication for cross-country differences in the concentration of sexually transmitted infections is that countries might be identical in terms of population size and the profile of preferences over number of sexual partners, but exhibit large differences in infection prevalence. This is because the probability that the networks that realize in identical economies be identical is very small, given the large number of equilibria. Similarly, the concentration of information about a new product, idea or technology might vary widely across populations of firms/workers or professors/students, even if the latter have identical distributions of social relationships.

Our model also sheds light on how network configuration may affect long-run gender asymmetry in information concentration. An instance of information may be a new sex technology that can only be learned from a sexual partner, or a sexually transmitted disease. Theorem 5 reveals that information is equally prevalent among men and women in societies where there is more equality between the sexes, but more women than men are informed or infected in male-dominant societies. To illustrate, consider the economy described by Example 2. Following the diffusion of a random contagion shock, contagion will be equally prevalent among men and women in g_1 and g_2 (that is, $\mathcal{F}(g_1) = \mathcal{F}(g_2) = 0$), two stochastically stable networks under the first process, and its prevalence will be greater for women than men by 13.5 and 24.5 percentage points in g_4 and g_5, respectively (that is, $\mathcal{F}(g_4) = -\frac{54}{400} < 0$ and $\mathcal{F}(g_5) = -\frac{98}{400} < 0$), two stochastically stable networks under the second process. Applying this insight to HIV/AIDS, Pongou and Serrano (2013) argue that, even if men are initially more infected than women in a society as in network g_3 ($\mathcal{F}(g_3) = \frac{12}{400} > 0$), in the long run, men and women will be equally vulnerable in monogamous societies which are better described by our first stochastic process (i.e., relationships are harder to break the lower the number of partners of one's old partner), whereas women will be more vulnerable in polygynous societies which are better described by the second process.[20] It also seems important to note that the two processes might coexist in the same society, but in different groups or segments of the population. For example, polygyny has been practised among the Mormons in the United States (Becker (1974)), coexisting with monogamy, which is practised by a large majority of the country. Also, in developed countries, immigrants coming from polygynous cultures might tend to exhibit the behavioral outcome of the second process, whereas natives might tend to behave according to the first process. In a situation where the two processes coexist, the information will concentrate more among women than among men, especially in those sectors of society better described by the second process.

In our other applications, news about a sale in a particular store spread equally among the buyers and sellers in the first process, whereas the word-of-mouth is more prevalent among buyers in the case of a thin or concentrated market. This logic might also help to understand the patterns of technological diffusion between rich and poor countries. A shock affecting work conditions in a firm extends equally to both sides if the market is composed of many small firms, while the information travels more among the workers when few firms employ all of them. Finally, in the faculty-students application, new ideas spread among students more easily in the model where a few professors are perceived to be of high quality, drawing large numbers of advisees.

5.3 Related Theoretical Literature

Aumann and Myerson (1988) and Jackson and Wolinsky (1996) pioneered the study of endogenous formation of links among agents. Aumann and Myerson (1988) examine a two-stage game. In its first stage, players form bilateral links resulting in a communication and cooperative structure, to which the Myerson value (Myerson (1977)) is applied in the second stage.[21] Jackson and Wolinsky (1996) introduce a framework for the study of the stability of networks among self-interested individuals. They develop a notion of pairwise stability of networks, and analyze its relationship with efficiency.[22]

[20] Morris and Kretzschmar (1997) also use a network model to study the differential effects of serial monogamy and concurrent partnerships on the spread of HIV/AIDS, but they do not address the question of its gender gap prevalence.

[21] For extensions and variants, see Dutta, van den Nouweland and Tijs (1996), and Slikker and van den Nouweland (2001a, b).

[22] Other studies on strategic network formation include Dutta and Mutuswami (1997), Bala and Goyal (2000), Watts (2001), Jackson and Watts (2002), Jackson and van den Nouweland (2005), Page, Wooders and Kamat (2005), Dutta, Ghosal and Ray (2005), Bloch and Jackson (2007). For authoritative monographs on networks, see Vega-Redondo (2007), Jackson (2008), and

As for dynamics within two-sided markets, the type of dynamics in which at each period, a pair of individuals can form and sever links goes back to Roth and Vande Vate (1990). More recently, several papers have also studied the dynamics of network formation using the notion of stochastic stability. Some of these papers include Jackson and Watts (2002) and Feri (2007). Although we also use stochastic stability as a "solution concept", our models, their interpretations, and their applications (to contagion in bipartite settings) differ from those in these studies.

Our paper also studies the endogenous formation of links, but there are some significant differences with previous work. First, our notion of pairwise stability allows for simultaneous link formation and severance and therefore differs from pairwise stability à la Jackson and Wolinsky (1996). Second, our focus is confined to networks where agents only decide the number of partners they desire, yielding complete characterizations given our assumptions. And third, our dynamic analysis rests on the notion that different transitions in link formation or severance have different probabilities (the different likelihood of our neutral actions), as opposed to uniform mistakes as is customary in the literature (see, e.g., Jackson and Watts (2002)).

Another distinctive feature of our model is that we avoid the standard coordination problem by looking at a continuous problem rather than a discrete one. Agents maximize in a continuous way their utility function to determine their optimal number of partners. A similar approach is adopted in Cabrales, Calvó-Armengol and Zenou (2011).[23] As in this study, agents in our model do not direct their links but decide the number of partners. What is key is the fact that the link formation process is not equivalent to elaborating a nominal list of intended relationships, as is the case in the literature on network formation. Network formation is therefore not the result of an earmarked socialization process, which enables us to totally characterize the pairwise stable matchings, something that has proved rather difficult in the standard framework.

Furthermore, compared to the dynamic network formation literature, our analysis innovates in that individuals do not form links at random as it has often been assumed (see, e.g., the preferential attachment model *à la* Jackson and Rogers (2007a)), but choose links that maximize their myopic utility. Similarly, individuals do not delete links at random but in a strategic way. Indirect links, however, do not matter in our model, as *a priori* agents may not even know their partners' other partners in certain applications, which enables a clean though not trivial characterization of long-run equilibria. Our results on pairwise stability also relate to the literature on stability in many-to-many matching markets (e.g., Echenique and Oviedo (2006)).

Finally, our paper also connects with the literature on social influence, social learning and contagion (see, e.g., Jackson and Rogers (2007b), Jackson and Yariv (2007), Lopez-Pintado (2008), Young (2009)). The different approaches used in these studies to analyze diffusion generally assume a connectivity distribution of the population, and/or a payoff function whose arguments include an individual's and her neighbors' choice of a certain behavior, and often rely on mean-field approximation theory to identify equilibria. Each

Easley and Kleinberg (2010).

[23] Several other papers have studied link formation based on utility considerations (see, e.g., Snijers (2001), and Staudigl (2011)).

individual faces the choice of adopting a certain behavior, such as buying a new product or not, and this behavior spreads as it is adopted. Our model differs in that it mostly studies "information transmission", not "information adoption." Distinguishing between the two notions is important. Within our framework, an agent who receives information about, say a new product, idea, or technology, communicates it to her friends, but we do not pose the receiver's choice problem. Also, in our application to sexual networks, an agent who is infected with a virus that spreads through sex infects his/her sexual partners; the latter do not make the choice of becoming infected, and the former may not even be aware of his/her status (in this sense, we are closer to the literature on epidemiological contagion (see, e.g., Pastor-Satorras and Vespignani (2000, 2001)). We also note, as remarked by Young (2009), that most papers on social diffusion assume, unlike we do, infinite populations and purely random meetings between individuals.

6 Conclusion

We view our contribution as three-fold. First, we have provided a full characterization of static equilibria (pairwise stable networks) in two-sided economies in which agents derive utility from the number of partners they have. Second, we have proposed a dynamic model of network formation in this class of economies. Under our general assumptions, we have characterized long-run equilibria (steady state networks), which coincide exactly with those networks that have the pairwise stability property. To gain predictive power, we resort to stochastic stability and prove results under two different cultures/perceptions of multiple partnerships. Third, we have applied the findings to understand the functioning and the patterns of relationships in several two-sided markets, including buyer/seller, rich/poor countries, employer/employee, dating, and faculty/student markets. In each of these markets, our stochastic processes potentially have different interpretations, providing a rationale for patterns of fragmentation, herding, and concentration that are usually observed. Also, our analysis rationalizes the fact that the number of sellers or employers who end up being active in trade is generally much smaller than the number of active buyers or employees. Furthermore, the application of the findings to contagion reveals that the configurations of long-run networks are such that the spread of any random information or contagion shock would (weakly) affect more *women* than *men*. Such a shock may also affect structurally identical economies differently.

A distinctive feature of the networks we have studied is that *a priori*, agents may not know their partners' other partners. In addition, they may not gain anything from these indirect connections. A natural extension of our analysis will be to consider the case in which an individual's well-being is affected by indirect "invisible" links and their consequent externalities. Our basic framework should be amenable to this and other realistic extensions, once incomplete information is incorporated to the analysis.

References

Aumann, R. (1959): "Acceptable Points in General Cooperative N-Person Games," in A. W. Tucker and R. D. Luce, Eds. *Contributions to the Theory of Games*, Vol. IV. Princeton: Princeton University Press, 1959.

Aumann, R., and R. Myerson (1988): "Endogenous Formation of Links Between Players and Coalitions: An Application of the Shapley Value," in A. Roth, Ed. *The Shapley Value*, Cambridge University Press, Cambridge, 1988.

Bala, V., and S. Goyal (2000a): "A Non-cooperative Model of Network Formation," *Econometrica* 68, 1181-1230.

Banerjee, A.V. (1992): "A Simple Model of Herd Behavior," *Quarterly Journal of Economics* 107, 797–817.

Becker, G.S. (1974): "A theory of Marriage: Part II," *Journal of Political Economy*, 82(2), S11-S26.

Bergin J, and B.L. Lipman (1996): "Evolution with state-dependent mutations," *Econometrica* 64, 943–956.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992): "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," *Journal of Political Economy* 100, 992-1026

Bloch, F., and M. Jackson (2007): "The Formation of Networks with Transfers Among Players," *Journal of Economic Theory* 133, 83-110.

Cabrales, A., A. Calvó-Armengol and Y. Zenou (2011): "Social Interactions and Spillovers: Incentives, Segregation and Topology," *Games and Economic Behavior* 72, 339-360.

Caldwell, J.C. (1976): "Marriage, the Family and Fertility in Sub-Saharan Africa with Special Reference to Research Programmes in Ghana and Nigeria," in S.A. Huzayyin and G. Acsádi, Eds. *Family and Marriage in some African and Asiatic Countries*, Research Monograph Series no. 6. Cairo: Cairo Demographic Centre.

Caldwell, J.C., P. Caldwell, and I.O. Orubuloye (1992): "The Family and Sexual Networking in Sub-Saharan Africa: Historical Regional Differences and Present-Day Implications," *Population Studies* 46, 385–410.

Caldwell, P. (1976): "Issues of Marriage and Marital Change: Tropical Africa and the Middle East," in S.A. Huzayyin and G. Acsádi, Eds. *Family and Marriage in some African and Asiatic Countries*, Research Monograph Series no. 6. Cairo: Cairo Demographic Centre.

Dutta, B., S. Ghosal and D. Ray (2005): "Farsighted Network Formation," *Journal of Economic Theory* 122, 143-164.

Dutta, B., and S. Mutuswami (1997): "Stable Networks," *Journal of Economic Theory* 76, 322-344.

Dutta, B., A. van den Nouweland, and S. Tijs (1995): "Link Formation in Cooperative Situations," *International Journal of Game Theory* 27, 245-256.

Easley, D. and J. Kleinberg (2010). *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*. Cambridge University Press.

Echenique, F., and J. Oviedo (2006): "A Theory of Stability in Many-to-Many Matching Markets," *Theoretical Economics* 1, 233-73.

Feri, F. (2007): "Stochastic stability in networks with decay," *Journal of Economic Theory* 135, 442-457.

Foster, D. P., and H. P. Young (1990): "Stochastic Evolutionary Game Dynamics," *Theoretical Population Biology* 38, 219-232.

Freidlin, M., and A. Wentzell (1984): *Random Perturbations of Dynamical Systems*, New York: Springer-Verlag.

Gale, D., and L. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly* 69, 9-15.

Jackson, M. O. (2008): *Social and Economic Networks*, Princeton: Princeton University Press.

Jackson, M.O, and B.W. Rogers (2007a): "Meeting Strangers and Friends of Friends: How Random are Social Networks?," *American Economic Review* 97, 890-915.

Jackson, M. O., and B.W. Rogers (2007b): "Relating Network Structure to Diffusion Properties Through Stochastic Dominance," *The B.E. Press Journal of Theoretical Economics* 7, 1-13.

Jackson, M.O., and A. van den Nouweland (2005): "Strongly Stable Networks," *Games and Economic Behavior* 51, 420-444.

Jackson, M.O., and A. Watts (2002): "The Evolution of Social and Economic Networks," *Journal of Economic Theory* 106, 265-295.

Jackson, M.O., and A. Wolinsky (1996): "A Strategic Model of Economic and Social Networks," *Journal of Economic Theory* 71, 44 74.

Jackson, M.O., and L. Yariv (2007): "Diffusion of Behavior and Equilibrium Properties in Network Games," *American Economic Review* 42, 92-98.

Kandori, M., G. Mailath, and R. Rob (1993): "Learning, Mutations and Long Run Equilibria in Games," *Econometrica* 61, 29-56.

Lopez-Pintado, D. (2008): "Contagion in Complex Networks," *Games and Economic Behavior* 62, 573-590.

Morris, M., M. Kretzschmar (1997): "Concurrent Partnerships and the Spread of HIV," *AIDS* 11, 641–648.

Myerson, R. (1977): "Graphs and Cooperation in Games," *Mathematics of Operations Research* 2, 225-229.

Page Jr., F.H., M.H. Wooders, and S. Kamat (2005): "Networks and Farsighted Stability," *Journal of Economic Theory* 120, 257-269.

Pastor-Satorras, R., and A. Vespignani (2000): "Epidemic Spreading in Scale-Free Networks," *Physical Review Letters* 86, 3200-3203.

Pastor-Satorras, R., and A. Vespignani (2001): "Epidemic Dynamics and Endemic States in Complex Networks," *Physical Review E* 63, 066-117.

Pongou, R. (2010): *The Economics of Fidelity in Network Formation,* PhD Dissertation, Department of Economics, Brown University (http://gradworks.umi.com/34/30/3430073.html).

Pongou, R., and R. Serrano (2009): "A Dynamic Theory of Fidelity Networks with an Application to the Spread of HIV/AIDS," Working Paper, Brown University.

Pongou, R., and R. Serrano (2013): "Fidelity Networks and Long-Run Trends in HIV/AIDS Gender Gaps," *American Economic Review* (Papers and Proceedings) 103(3), 298-302.

Quale, G.R. (1992): Families in Context: A World History of Population. Westport, CT: Greenwood.

Roth, A.E. (2007): "Repugnance as a Constraint on Markets," *Journal of Economic Perspectives* 21, 37-58.

Roth, A., and J.H. Vande Vate (1990): "Random Paths to Stability in Two-Sided Matching," *Econometrica* 58, 1475-1480

Schoen, R., W. Urton, K. Woodrow, and J. Baj (1985): "Marriage and Divorce in twentieth Century America Cohorts," *Demography* 22, 101-114.

Slikker, M., and A. van den Nouweland (2001a): *Social and Economic Networks in Cooperative Game Theory*, Kluwer Academic Publishers.

Slikker, M., and A. van den Nouweland (2001b): "A One-stage Model of Link Formation and Payoff Division," *Games Economic Behavior* 34, 153-175.

Snijders, T. (2001): "The Statistical Evaluation of Social Network Dynamics," *Sociological Methodology* 31, 361-395.

Staudigl, M. (2011): "Potential Games in Volatile Environments," *Games and Economic Behavior* 72, 271-287.

Tertilt, M. (2005): "Polygyny, Fertility, and Savings," *Journal of Political Economy* 113, 1341-1371

Vega-Redondo, F. (2007): *Complex Social Networks*, Econometric Society Monograph: Cambridge University Press.

Watts A. (2001): "A Dynamic Model of Network Formation," *Games and Economic Behavior* 34, 331-341.

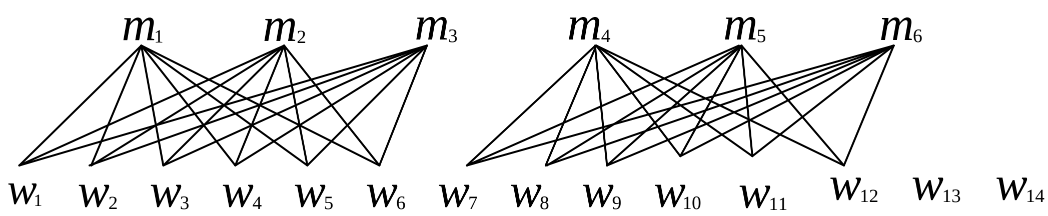
Young, H. P. (1993): "The Evolution of Conventions," *Econometrica* 61, 57-84.

Young, H. P. (1998): *Individual Strategy and Social Structure: an Evolutionary Theory of Institutions*, Princeton University Press.

Young, H.P. (2009): "Innovation Diffusion in Heterogeneous Populations: Contagion, Social Influence, and Social Learning," *American Economic Review* 99 (5), 1899-1924.

Figures of Example 1

Figure 1-1



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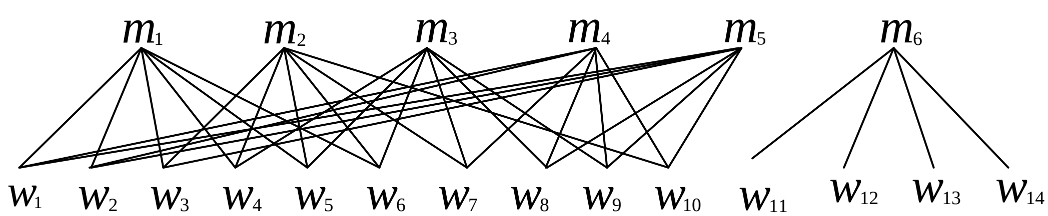

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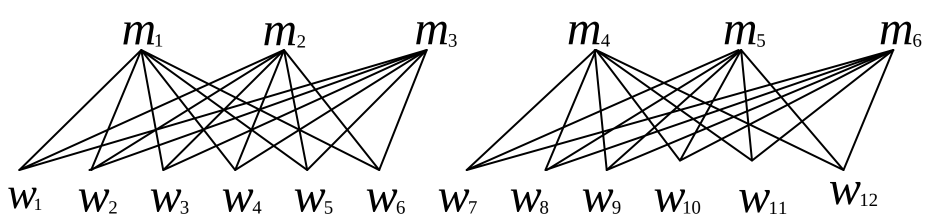

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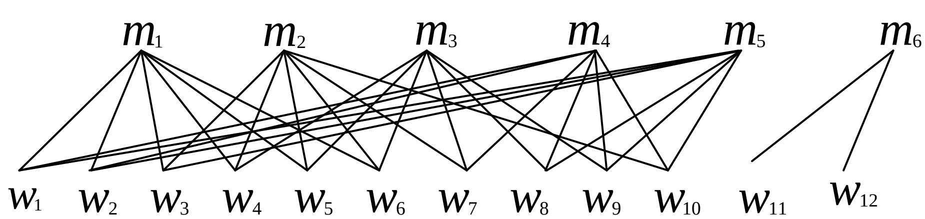

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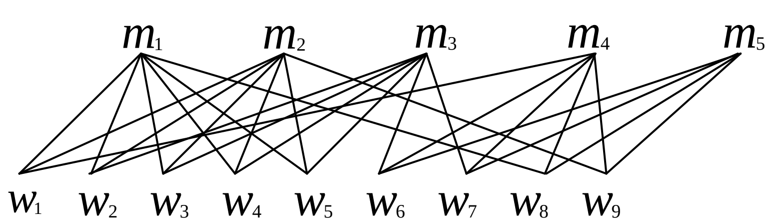

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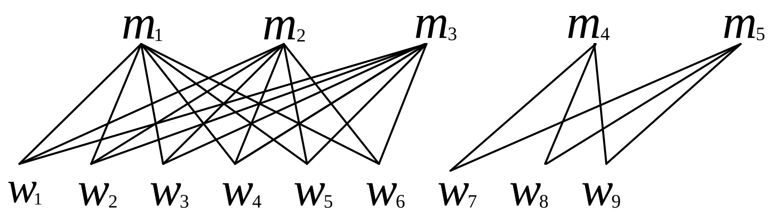

Figures of Example 2

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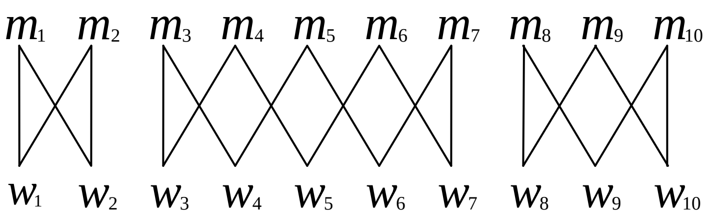

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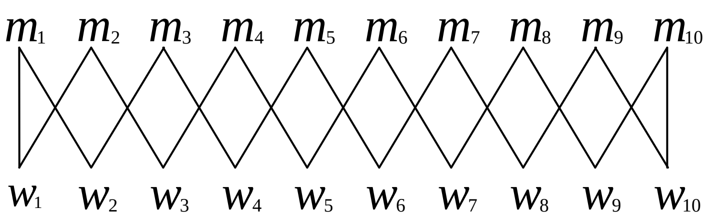

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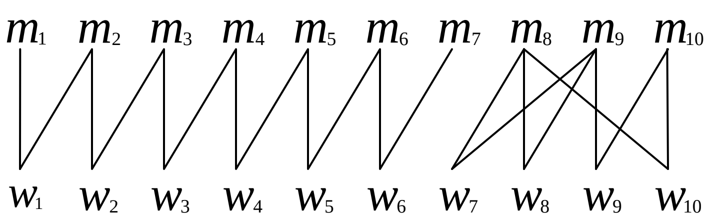

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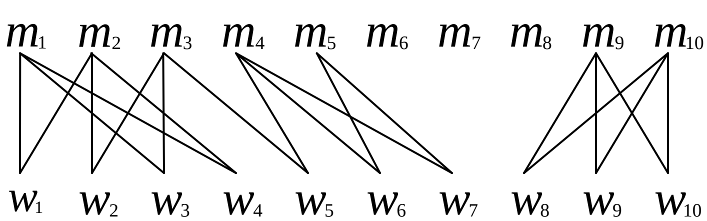

Figure 2-5